GEOMETRIC APPROACH TO HALL ALGEBRA OF REPRESENTATIONS OF QUIVERS OVER LOCAL RINGS

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Abstract. The category of representations of quivers over local rings $R = k[t]/(t^n)$ is no longer hereditary. The Hall algebra defined on this category doesn't have a well defined coalgebra structure. In the present paper, the full subcategory of this category, whose objects are the modules assigning a free R-module to each vertex, is considered. This full subcategory is an exact category on which the Ringel-Hall algebra is well defined. A geometric realization of the composition subalgebra of this Hall algebra is given under the framework of Lusztig's geometric setting. Moreover, the canonical basis and a monomial basis of this subalgebra are constructed by using perverse sheaves. This generalizes Lusztig's result about the geometric realization of quantum enveloping algebra. As a byproduct, the relation between this subalgebra and quantum generalized Kac-Moody algebra is obtained.
Key Words: Quantum generalized Kac-Moody algebra; Hall algebra; representations of quivers; local ring; exact category; perverse sheaves

1. Introduction

Given a quiver Γ, one may consider the representations of Γ over a commutative ring, R, namely, assigning an R-module to each vertex and an R-linear map to each arrow. When $R = k$ is a field, a representation of Γ assigns a vector space to each vertex and a k-linear map to each arrow. In the case that $k = \mathbb{F}_q$ is a finite field, one may define a multiplication on the free abelian group generated by the isomorphism classes of $k\Gamma$-modules by counting filtrations of certain submodules, where $k\Gamma$ is the path algebra of the quiver Γ. The number of filtrations is called a Hall number, and the ring obtained in this way is called the Hall algebra, $\mathcal{H}(k\Gamma)$, see [27].

Fix a Dynkin quiver Γ. For given representations M, N, E, the Hall numbers $F^E_{M,N}$ depend on the cardinality, q, of k. Precisely, one may find polynomials of q as structure constants of the Hall algebra $\mathcal{H}(k\Gamma)$. Such polynomials are called Hall polynomials, see [28]. Thus the free $\mathbb{Z}[q]$-module, $\mathcal{H}(k\Gamma)$, is well defined and is the generic Hall algebra.

In the case when k is a finite field and Γ is a Dynkin quiver, Ringel shows that $\mathcal{H}(k\Gamma)$ is generated by simple modules. Moreover, after twisting by using the Euler character on the Grothendick group, $K_0(k\Gamma)$, one may obtain the twisted Hall algebra $\mathcal{H}_\star(k\Gamma)$ which is isomorphic to U_q^+ as an algebra, see [28, 29]. Here U_q^+ is the positive part of the quantum enveloping algebra of the Lie algebra corresponding to the given quiver. Later on, Green, in [7], defined a coalgebra structure on $\mathcal{H}_\star(k\Gamma)$. Additionally Xiao, in [31], defined the antipode of $\mathcal{H}_\star(k\Gamma)$. This makes $\mathcal{H}_\star(k\Gamma)$ a Hopf algebra.

In [20, 16, 17, 18, 23, 21], Lusztig gives a geometric realization of U_q^+ and constructs its canonical basis by using simple perverse sheaves on the representation space of the

corresponding quiver, also see [13]. Later on, this work is generalized to the affine case, see [11, 12, 19] for more information.

In the present paper, we consider free representations (see definition in section 2.1) of loop-free quivers over the local ring $R = \mathbb{F}_q[t]/(t^n)$. In this case, the category is not an abelian category anymore, but rather an exact category. The representations of quivers over the local ring $R = k[t]/(t^n)$ have a close relationship with the representations over field k. The simplest example is the quiver Γ with one vertex and no arrows. In this example, the category of representations of the quiver Γ over $R = k[t]/(t^n)$ is the category of R-module. This category is equivalent to the category of nilpotent representations of the Jordan quiver over k. In [10], Kang and Schiffmann give a geometric realization of the positive part of the quantized enveloping algebra of a generalized Kac-Moody Lie algebra by using quivers with multiple loops. So, one may expect to approach Kang and Schiffimann's result through the representations of quivers over the local ring $R = k[t]/(t^n)$. Another motivation of this paper is Lusztig's paper [22]. He considers the representations of reductive groups over the finite ring $R = \mathbb{F}_q[t]/(t^n)$. This is a full subcategory of the category of representations of the Jordan quiver over R.

Hubery, in [9], defined a Hall algebra on an exact category, but, in this case, it is not clear that if there exists a coalgebra structure for this algebra although the homological dimension of the category is 1, see Remark 1.11 (iii) in [30]. The category of free representations of quivers over local ring $R = k[t]/(t^n)$ servers as a counterexample, see Section 3.3.2.

In the local ring $R = k[t]/(t^n)$ case, if we consider representation spaces under the framework of Lusztig's geometric setting, we can still study the geometric realization of the composition subalgebra. In this context, the main difference from the field case is that the flag varieties consisting of the filtration of graded free R-modules is not a projective variety. This causes the projection map along the flag variety to not be a proper map. Hence, the decomposition theorem in [1] can not be directly applied here. Fortunately, the projection map is still nice enough because we can decompose the projection map into a composition of a proper map and a vector bundle. Lusztig's parabolic induction functor which gives Hall multiplication of this algebra is well-defined in this case, but the restriction functor, which gives coalgebra structure in the field case, is not well-defined in this case (see Section 4.2). Moreover, we can use simple perverse sheaves to construct the canonical basis of the composition subalgebra, and the relationship between this subalgebra and the quantum generalized Kac-Moody algebra is discussed. This gives Kang and Schiffmann's result in the special case that the charge is $m_i = 1$ for all i. In this special case, the canonical basis can be constructed by the simple perverse sheaves but not semisimple perverse sheaves. This refines their result.

This paper is organized as follows. In Section 2, we briefly review the theory of perverse sheaves, most of which can be found in [1, 5, 23] and will be used in Section 4 and 5. In Section 3, we deform Hubery's definition of Hall algebra on the exact category. In Section 4, we study the geometric property of flag varieties on R-modules and perverse sheaves on the representation space. The Grothendieck group $\mathcal{K}$ of such perverse sheaves with the induction functor gives the geometric realization of the composition Hall algebra. In addition, this also gives an example of perverse sheaves on Jet schemes. Additionally, we obtain the canonical basis and a monomial basis of the composition subalgebra. In Section 5, the relations among $\mathcal{K}$, the composition subalgebra and generalized Kac-Moody algebras

are obtained. In addition, the canonical basis and a monomial basis of the composition subalgebra are given.

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2. Preliminary

In this section, we will quickly review the theory of perverse sheaves. For reference, we refer to Chapter 8 in [23]. The reader can also find these in [1, 5, 4].

Let k be the algebraic closure of $\mathbb{F}_q$, and let all algebraic varieties be over k and of finite type separable.

2.1. **Perverse sheaves.** Let X be an algebraic variety. Denote by $\mathcal{D}(X) = \mathcal{D}^b_c(X)$ the bounded derived category of $\overline{\mathbb{Q}}_l$-constructible sheaves. Here l is a fixed prime number which is invertible in k, and $\overline{\mathbb{Q}}_l$ is the algebraic closure of the field $\mathbb{Q}_l$ of l-adic numbers. Objects of $\mathcal{D}(X)$ are referred to as complexes. For a complex $K \in \mathcal{D}(X)$, denote by $\mathcal{H}^n(K)$ the n-th cohomology sheaf of K. For any integer j, let $[j] : \mathcal{D}(X) \to \mathcal{D}(X)$ be the shift functor which satisfies $\mathcal{H}^n(K[j]) = \mathcal{H}^{n+j}(K)$.

Let $f : X \to Y$ be a morphism of algebraic varieties. There are functors $f^* : \mathcal{D}(Y) \to \mathcal{D}(X)$, $f_* : \mathcal{D}(X) \to \mathcal{D}(Y)$, $f_! : \mathcal{D}(X) \to \mathcal{D}(Y)$ (direct image with compact support), and $f^! : \mathcal{D}(Y) \to \mathcal{D}(X)$.

Let $p_X : X \to \{\mathrm{pt}\}$ be the morphism from an algebraic variety X to a point. Denote by $\mathbf{1} = \mathbf{1}_X$ the $\overline{\mathbb{Q}}_l$-constant sheaf on X. The complex $\dot{\omega}_X = (p_X)^!(\mathbf{1}_{\mathrm{pt}})$ is called the dualizing complex on X. And $\mathbb{D}K = R\mathcal{H}om(K, \dot{\omega}_X) \in \mathcal{D}(X)$ is called the Verdier dual of $K \in \mathcal{D}(X)$.

In this paper, the perversity refers to the middle perversity. To define perverse sheaves, we first introduce two full subcategories which define a t-structure on $\mathcal{D}(X)$. An object $K \in \mathcal{D}(X)$ is said to satisfy

$$\begin{cases} (1) \quad \textit{support condition} \text{ if } \dim \mathrm{Supp}\mathcal{H}^n(K) \leq -n, \forall n; \\ (2) \quad \textit{cosupport condition} \text{ if } \dim \mathrm{Supp}\mathcal{H}^n(\mathbb{D}K) \leq -n, \forall n. \end{cases}$$

Let $\mathcal{D}(X)^{\leq 0}$ be the full subcategory of $\mathcal{D}(X)$ whose objects satisfy support condition. In particular, $\mathcal{H}^n(K) = 0$ for $n > 0$. Let $\mathcal{D}(X)^{\geq 0}$ be the full subcategory of $\mathcal{D}(X)$ whose objects satisfy cosupport condition. Then $(\mathcal{D}(X)^{\leq 0}, \mathcal{D}(X)^{\geq 0})$ defines a t-structure on $\mathcal{D}(X)$.

Let $\mathcal{M}(X)$ be the full subcategory of $\mathcal{D}(X)$ whose objects are in $\mathcal{D}(X)^{\leq 0} \bigcap \mathcal{D}(X)^{\geq 0}$. The objects of $\mathcal{M}(X)$ are called perverse sheaves on X. $\mathcal{M}(X)$ is the heart of the t-structure and is actually an abelian category in which all objects have finite length. The simple objects of $\mathcal{M}(X)$ are given by the Deligne-Goresky-Macpherson intersection cohomology complexes corresponding to various smooth irreducible subvarieties of X and to irreducible local systems on them.

Let $\tau_{\leq 0}$ (resp. $\tau_{\geq 0}$) : $\mathcal{D}(X) \to \mathcal{D}(X)$ be the truncation functor. Then we have a functor

$$\begin{aligned} {}^pH^0 : \mathcal{D}(X) &\to \mathcal{M}(X) \\ K &\mapsto \tau_{\geq 0}\tau_{\leq 0}K. \end{aligned}$$

Define the perverse cohomology functor ${}^pH^n : \mathcal{D}(X) \to \mathcal{M}(X)$ as ${}^pH^n(K) = {}^pH^0(K[n])$.

A complex $K \in \mathcal{D}(X)$ is called semisimple if ${}^pH^n(K)$ is semisimple in $\mathcal{M}(X)$ for all n and K is isomorphic to $\oplus_n {}^pH^n(K)[-n]$ in $\mathcal{D}(X)$.

For any integer n, denote by $\mathcal{M}(X)[n]$ the full subcategory of $\mathcal{D}(X)$ whose objects are of the form $K[n]$ for some $K \in \mathcal{M}(X)$.

2.2. **Properties of functors.** Let $f : X \to Y$ be a morphism of algebraic varieties. The functors $f^*, f_!, f^!, f_*, [j]$ and the Verdier dual $\mathbb{D}$ satisfy the following properties.

2.2.1. *Adjunction.* If $f : X \to Y$, then (f^*, f_*) and $(f_!, f^!)$ are adjoint pairs. i.e. for any $A \in \mathcal{D}(X),\ B \in \mathcal{D}(Y)$,

(1) $\mathrm{Hom}_{\mathcal{D}(X)}(f^*B, A) = \mathrm{Hom}_{\mathcal{D}(Y)}(B, f_*A)$;
(2) $\mathrm{Hom}_{\mathcal{D}(Y)}(f_!A, B) = \mathrm{Hom}_{\mathcal{D}(X)}(A, f^!B)$.

2.2.2. *Pull back.* If $f : X \to Y$ is smooth with connected fibers of dimension d, let $\widetilde{f} = f^* \circ [d]$, then we have the following properties,

(1) $f^! = f^*[2d]$ and $\mathbb{D}f^*(B) = f^!(\mathbb{D}B)$. (We will ignore the Tate twist.)
(2) $K \in \mathcal{D}(Y)^{\leq 0} \Leftrightarrow \widetilde{f}K \in \mathcal{D}(X)^{\leq 0}$.
(3) $K \in \mathcal{D}(Y)^{\geq 0} \Leftrightarrow \widetilde{f}K \in \mathcal{D}(X)^{\geq 0}$.
(4) $K \in \mathcal{M}(Y) \Leftrightarrow \widetilde{f}K \in \mathcal{M}(X)$.
(5) ${}^pH^i(\widetilde{f}K) = \widetilde{f}({}^pH^i(K))$.
(6) If $K \in \mathcal{D}Y^{\leq 0}$ and $K' \in \mathcal{D}Y^{\geq 0}$, then
$$\mathrm{Hom}_{\mathcal{D}(Y)}(K, K') = \mathrm{Hom}_{\mathcal{D}(X)}(\widetilde{f}K, \widetilde{f}K').$$
(7) $\widetilde{f} : \mathcal{M}(Y) \to \mathcal{M}(X)$ is a fully faithful functor.
(8) If $K \in \mathcal{M}(Y)$ and $K' \in \mathcal{M}(X)$ is a subquotient of $\widetilde{f}K \in \mathcal{M}(X)$, then K' is isomorphic to $\widetilde{f}K_1$ for some $K_1 \in \mathcal{M}(Y)$.

2.2.3. *Pushforward and decomposition.*

(1) If $f : X \to Y$ is a proper morphism, then $f_* = f_!$ and $f_!(\mathbb{D}A) = \mathbb{D}f_!(A)$.
(2) If $f : X \to Y$ is a proper morphism with X smooth, then $f_!(\mathbf{1}) \in \mathcal{D}(Y)$ is a semisimple complex.
(3) Let $f : X \to Y$ be a morphism of varieties. If there is a partition $X = X_0 \cup X_1 \cup \cdots \cup X_m$ of locally closed subvarieties, such that $X_{\leq j} = X_0 \cup \cdots \cup X_j$ is closed for $j = 0, \cdots, m$, and for each j there are morphisms $X_j \xrightarrow{f_j} Z_j \xrightarrow{f_j'} Y_j$, such that Z_j is smooth, f_j is a vector bundle, f_j' is proper and $f_j'f_j = f|_{X_j}$, then $f_!(\mathbf{1}) \in \mathcal{D}(Y)$ is a semisimple complex. Additionally, for any n and j, there is a canonical exact sequence:
$$0 \longrightarrow {}^pH^n(f_j)_!\mathbf{1} \longrightarrow {}^pH^n(f_{\leq j})_!\mathbf{1} \longrightarrow {}^pH^n(f_{\leq j-1})_!\mathbf{1} \longrightarrow 0,$$

where $f_{\leq j}$ and f_j are the restrictions of f.

(4) Let X be an algebraic variety, U be an open subset of X, and Z be the complement of U in X. Let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be the inclusions. For any $K \in \mathcal{D}(X)$, there is a canonical distinguished triangle in $\mathcal{D}(X)$,

$$j_!j^*K \longrightarrow K \longrightarrow i_!i^*K \xrightarrow{[1]} .$$

If $f : X \to Y$, then we have a canonical distinguished triangle in $\mathcal{D}(Y)$,

$$f_!j_!j^*K \longrightarrow f_!K \longrightarrow f_!i_!i^*K \xrightarrow{[1]} .$$

2.2.4. *Base change.* If

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "r"'] & Y \arrow[d, "s"] \\ Z \arrow[r, "g"] & W \end{tikzcd}$$

is a cartesian square and s is proper (resp. g is smooth), then

$$r_!f^* = g^*s_! : \mathcal{D}(Y) \to \mathcal{D}(Z).$$

Usually it is called a proper (resp. smooth) base change.

2.2.5. *Projection formula.* Let $f : X \to Y$ be a morphism of varieties. $C \in \mathcal{D}(X)$ and $K \in \mathcal{D}(Y)$ are constructible complexes, then

$$K \otimes f_!C \simeq f_!(f^*K \otimes C).$$

2.3. G**-equivariant complexes.** Let $m : G \times X \to X$ be an action of a connected algebraic group G on X and $\pi : G \times X \to X$ be the second projection. Both maps are smooth with connected fiber of $\dim G$. A perverse sheaf K on X is said to be G-equivariant if the perverse sheaves $\pi^*K[\mathrm{dimG}]$ and $m^*K[\mathrm{dimG}]$ are isomorphic. More generally, a complex $K \in \mathcal{M}(X)[n]$ is said to be G-equivariant if the perverse sheaf $K[-n]$ is G-equivariant. Denote by $\mathcal{M}_G(X)$ the full subcategory of $\mathcal{M}(X)$ whose objects are the G-equivariant perverse sheaves on X. More generally, denote by $\mathcal{M}_G(X)[n]$ the full subcategory of $\mathcal{M}(X)[n]$ whose objects are of the form $K[n]$ with $K \in \mathcal{M}_G(X)$.

(1) If $A \in \mathcal{M}_G(X)$, and $B \in \mathcal{M}(X)$ is a subquotient of A, then $B \in \mathcal{M}_G(X)$.
(2) Assume $f : X \to Y$ is a G-equivariant morphism. If $K \in \mathcal{M}_G(Y)$, then ${}^pH^n(f^*K) \in \mathcal{M}_G(X)$ for all n. If $K' \in \mathcal{M}_G(X)$, then ${}^pH^n(f_!K') \in \mathcal{M}_G(Y)$ for all n.
(3) Assume that $f : X \to Y$ is a locally trivial principal G-bundle. Let $d = \dim(G)$. If $K \in \mathcal{M}(Y)[n+d]$, then $f^*K \in \mathcal{M}_G(X)[n]$. Furthermore, the functor $f^* : \mathcal{M}(Y)[n+d] \to \mathcal{M}_G(X)[n]$ defines an equivalence of categories. The inverse $f_\flat : \mathcal{M}_G(X)[n] \to \mathcal{M}(Y)[n+d]$ is given by $f_\flat(K) = H^{-n-d}(f_*K)[n+d]$.

2.4. **Fourier Deligne transformations.** The Artin-Schreier covering $k \to k$ sending x to $x^p - x$ has $\mathbb{F}_p$ as a group of covering transformations. Hence any non-trivial character $\phi : \mathbb{F}_p \to \overline{\mathbb{Q}}_l^*$ gives rise to a $\overline{\mathbb{Q}}_l$ local system, $\mathcal{E}$, of rank 1 on k. Let $T : X \to k$ be any morphism of algebraic varieties. Then $\mathcal{L}_T := T^*\mathcal{E}$ is a local system of rank 1 on X.

Now let $E \to X$ and $E' \to X$ be two vector bundles of constant fiber dimension d over X. Let $T : E \times_X E' \to k$ be a bilinear map which defines a duality between the two vector bundles. Consider the following diagram,

$$E \xleftarrow{s} E \times_X E' \xrightarrow{t} E' ,$$

where s, t are projection maps. Define

$$\begin{aligned} \Phi : \mathcal{D}(E) &\to \mathcal{D}(E') \\ K &\mapsto t_!(s^*K \otimes \mathcal{L}_T)[d]. \end{aligned}$$

This functor is called a Fourier-Deligne transform.

If we interchange the roles of E, E', then we have another Fourier-Deligne transform, by abuse of notation, we still denote it by $\Phi : \mathcal{D}(E') \to \mathcal{D}(E)$. Moreover, we have $\Phi(\Phi(K)) = j^*K$ for any $K \in \mathcal{D}(E)$, where $j : E \to E$ is multiplication by -1 on each fiber of E.

If we restricts Φ to perverse sheaves, then $\Phi|_{\mathcal{M}(E)} : \mathcal{M}(E) \simeq \mathcal{M}(E')$. Moreover ${}^pH^n(\Phi(K)) = \Phi({}^pH^n K)$ for $K \in \mathcal{D}(E)$.

We will use the following two properties in Section 4 (see [23]).

(1) Let A (resp. A') be an object of $\mathcal{D}(E)$ (resp. $\mathcal{D}(E')$). Let u (resp. $u', \dot{u}$) be the map of E (resp. $E', E \times_X E'$) to the point. Then we have

$$u_!(A \otimes \Phi(A')) = \dot{u}_!(s^*A \otimes t^*A' \otimes \mathcal{L}_T[d]) = u'_!(\Phi(A) \otimes A').$$

(2) Let $T : k^n \to k$ be a non-constant affine linear function. Let $u : k^n \to \{\mathrm{pt}\}$. Then $u_!(\mathcal{L}_T) = 0$.

2.5. **Characteristic functions of complexes.** For the definition of characteristic function and its properties, we refer to [13, 5].

Let X be an algebraic variety over k. Let F be a Frobenius morphism of X and X^F be the set of fixed points by F. For any complex $\mathcal{F} \in D^b(X^F, \overline{\mathbb{Q}}_l)$, such that $F^*\mathcal{F} \simeq \mathcal{F}$, we choose for each such $\mathcal{F}$ an isomorphism $\phi_{\mathcal{F}}$. The characteristic function of $\mathcal{F}$ with respect to $\phi_{\mathcal{F}}$, denote by $\chi_{\mathcal{F},\phi_{\mathcal{F}}}$ can be defined as follows

$$\chi_{\mathcal{F},\phi_{\mathcal{F}}}(x) = Tr(\phi_{\mathcal{F},x} : \mathcal{F}_x \to \mathcal{F}_x), \ \forall \ x \in X^F,$$

where $\mathcal{F}_x$ is the stalk of $\mathcal{F}$ at x. We list some properties of characteristic functions in the following.

(1) If $f : X \to Y$ is a morphism defined over $\mathbb{F}_q$, and $\mathcal{F} \in D^b(X^F, \overline{\mathbb{Q}}_l)$, then for any $y \in Y^F$,

$$\chi_{f_!\mathcal{F},\phi_{\mathcal{F}}}(y) = \sum_{x \in f^{-1}(y)^F} \chi_{\mathcal{F},\phi_{\mathcal{F}}}(x).$$

(2) If $f : X \to Y$ is a morphism defined over $\mathbb{F}_q$, and $\mathcal{G} \in D^b(Y^F, \overline{\mathbb{Q}}_l)$, then for any $x \in X^F$,

$$\chi_{f^*\mathcal{G},\phi_{\mathcal{F}}}(x) = \chi_{\mathcal{G},\phi_{\mathcal{F}}}(f(x)).$$

3. Ringel Hall algebra

In this section, we fix $R = k[t]/(t^n)$ and consider R-free representations of loops free quivers. Denoted $Rep_R^f(\Gamma)$ the category consisting of all R-free representations of Γ. $Rep_R^f(\Gamma)$ is not an abelian category but rather an exact category. Hubery defines the Hall algebra over an exact category in [9]. Let $\mathcal{H}_R(\Gamma)$ be the Hall algebra on $Rep_R^f(\Gamma)$. One can ask if there exists a coalgebra structure on the Hall algebra over exact category. In general, this is not true (even the homological dimension of the exact category is 1). The category $Rep_R^f(Q)$ serves as a counterexample to that the Hall algebra on it has no coalgebra structure. We will give a geometric realization of the composition subalgebra of $\mathcal{H}_R(\Gamma)$.

3.1. **Exact category.** Let $\mathcal{A}$ be an additive category which is a full subcategory of an abelian category $\mathcal{B}$ and closed under extension in $\mathcal{B}$. Let $\mathcal{E}$ be a class of sequences

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{j} M'' \longrightarrow 0$$

in $\mathcal{A}$ which are exact in the abelian category $\mathcal{B}$. A map f is called an inflation (resp. a deflation) if it occurs as the map i (resp. j) of some members in $\mathcal{E}$. Inflations and deflations will be denoted by $M' \rightarrowtail M$ and $M \twoheadrightarrow M''$ respectively. The pair $M' \rightarrowtail M \twoheadrightarrow M''$ is called a conflation. The following is Quillen's definition of an exact category. See [3] for more properties of an exact category.

Definition 1. [26] An exact category is the additive category $\mathcal{A}$ equipped with a family $\mathcal{E}$ of the short exact sequences of $\mathcal{A}$, such that the following properties hold:

(i) Any sequence in $\mathcal{A}$ which is isomorphic to a sequence in $\mathcal{E}$ is in $\mathcal{E}$, and the split sequences in $\mathcal{A}$ are in $\mathcal{E}$.
(ii) The class of deflations is closed under composition and under base change by an arbitrary map in $\mathcal{A}$. Dually, the class of inflations is closed under composition and base change by an arbitrary map in $\mathcal{A}$.
(iii) Let $M \to M''$ be a map possessing a kernel in $\mathcal{A}$. If there exists a map $N \to M$ in $\mathcal{A}$ such that $N \to M \to M''$ is a deflation, then $M \to M''$ is a deflation. Dually, let $M' \to M$ be a map possessing a cokernel in $\mathcal{A}$. If there exists a map $M \to L$ in $\mathcal{A}$ such that $M' \to M \to L$ is an inflation, then $M' \to M$ is an inflation.

3.2. **Representation of quivers over commutative rings.** A *representation* (V, x) of $\Gamma = (I, H, s, t)$ over a commutative ring R is an I-graded R-module V together with a set $\{x_h\}_{h\in H}$ of R-linear transformations $x_h : V_{h'} \to V_{h''}$.

A *homomorphism* from one representation (V, x) to another representation (W, y) is a collection $\{g_i\}_{i\in I}$ of R-linear maps $g_i : V_i \to W_i$, such that $g_{h''}x_h = y_h g_{h'}$ for all $h \in H$. If all g_i are R-isomorphisms, (V, x) and (W, y) are said to be *isomorphic.*

Let $Rep_R(\Gamma)$ be the category of representations of Γ over R. $Rep_R(\Gamma)$ is an abelian category. If V is an I-graded free R-module, then the representation (V, x) is called an *R-free representation.* All such representations form a full subcategory $Rep_R^f(\Gamma)$ of $Rep_R(\Gamma)$. Unfortunately, this subcategory is not an abelian category anymore, but rather an exact category. In the following, all representations are assumed to be free over R. In this case, we can define the dimension vector $|V| := (\mathrm{Rank}_R(V_i))_{i\in I}$.

Lemma 1. *$Rep_R^f(\Gamma)$ is an exact category with homological dimension 1.*

Proof. It is easy to see $Rep^f_R(\Gamma)$ is an additive category. Let $\mathcal{E}$ be the set of all possible short exact sequences in $Rep^f_R(\Gamma)$. Then $Rep^f_R(\Gamma)$ with the class $\mathcal{E}$ is an exact category. In this case, an inflation is an injective map, such that the cokernel is an I-graded free R-module and a deflation map is a surjective map, such that the kernel is an I-graded free R-module.

Let $A = R\Gamma$. To show the homological dimension of $Rep^f_R(\Gamma)$ is 1, it is enough to show that sequence,

$$0 \longrightarrow \oplus_{\rho\in H} Ae_{\rho''} \otimes_R e_{\rho'} X \xrightarrow{f} \oplus_{i\in I} Ae_i \otimes_R e_i X \xrightarrow{g} X \longrightarrow 0 \tag{1}$$

is exact for any R-free left A-module X, where e_i is the trivial path for the vertex i, and $g(a\otimes x) = ax, f(a \otimes x) = a\rho \otimes x - a \otimes \rho x$.

In fact, the proof of exactness is the same as Crawley-Boevey's proof for the stand resolution in [2]. $\square$

Here the homological dimension 1 refers to $\mathrm{Ext}^n(X, Y)$ vanishing for all $n \geq 2$ and $X, Y \in \mathcal{A}$. See Chapter 6 in [6] for the definition of $\mathrm{Ext}^n(X, Y)$ in an exact category.

3.3. **Hall algebra over an exact category.** In this section, we first deform Hubery's definition of the Hall algebra over a finitary exact category, then give a counterexample to show that a coalgebra structure of the Hall algebra can not be obtained by twisting. We will always assume $\mathcal{A}$ is an exact category which is a full subcategory of an abelian category $\mathcal{B}$.

3.3.1. *The algebra structure.* Let $\mathcal{A}$ be a finitary and small exact category. Denote by W^L_{XY} the set of all conflations $Y \rightarrowtail L \twoheadrightarrow X$. The group $Aut(X) \times Aut(Y)$ acts on W^L_{XY} via:

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "\eta"] & L \arrow[r, "g"] \arrow[d, equal] & X \arrow[d, "\varepsilon"] \\
Y \arrow[r, "\overline{f}"] & L \arrow[r, "\overline{g}"] & X.
\end{tikzcd}$$

Denote by V^L_{XY} the quotient set of W^L_{XY} by the group $Aut(X) \times Aut(Y)$. Since f is an inflation and g is a deflation, this action is free. So

$$F^L_{XY} := |V^L_{XY}| = \frac{|W^L_{XY}|}{a_X a_Y},$$

where $a_X = |Aut(X)|$. The Ringle-Hall algebra $\mathcal{H}(\mathcal{A})$ is defined as the free $\mathbb{Z}$-module on the set of isomorphism classes of objects. By abuse of notation, we will write X for the isomorphism classes $[X]$, and use the numbers F^L_{XY} as the structure constants of multiplication. Define

$$X \circ Y := \sum_L F^L_{XY} L,$$

Theorem 1 ([9])**.** *The Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ of a finitary and small exact category $\mathcal{A}$ is an associate, unital algebra.*

If $\mathcal{A} = Rep_R^f(\Gamma)$, we want to deform the Ringel-Hall algebra $\mathcal{H}(Rep_R^f(\Gamma))$. Firstly, for $\alpha = (a_i)_{i\in I}, \beta = (b_i)_{i \in I}$, define

$$\langle \alpha, \beta \rangle := \sum_{i\in I} a_i b_i - \sum_{h \in H} a_{h'} b_{h''}. \tag{2}$$

It is easy to check this is a bilinear form on $\mathbb{N}^I$.

For any $X, Y \in Rep_R^f(\Gamma)$, define a deformed multiplication as

$$XY := q^{n\langle |X|, |Y| \rangle} X \circ Y.$$

Here $|X|$ is the dimension vector of X which is defined in last section.

Theorem 2. *$\mathcal{H}(Rep_R^f(\Gamma))$ equipped with the deformed multiplication is an associate, unital algebra.*

Proof. By Theorem 1, it is enough to prove that for any $\alpha, \beta, \gamma \in \mathbb{N}^I$,

$$\langle \alpha, \beta \rangle + \langle \alpha + \beta, \gamma \rangle = \langle \beta, \gamma \rangle + \langle \alpha, \beta + \gamma \rangle.$$

From the bilinearity of $\langle -, - \rangle$, both sides are equal to $\langle \alpha, \beta \rangle + \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$. $\square$

3.3.2. *The coalgebra structure.* Let $\Delta : \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{A})$ be the map as following,

$$\Delta(E) := \sum_{M,N} \langle M, N \rangle F_{MN}^E \frac{a_M a_N}{a_E} M \otimes N, \tag{3}$$

where M, N run through all conflations $M \rightarrowtail E \twoheadrightarrow N$. If one defines the twisted multiplication on $\mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{A})$ to be

$$(A \otimes B) \cdot (C \otimes D) := q^{\frac{n}{2}(\langle B, C \rangle + \langle C, B \rangle)} AC \otimes BD, \tag{4}$$

then as Green shows, in [7], the map Δ defined in (3) is an algebra homomorphism with respect to this twisted multiplication on $\mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{A})$ when $\mathcal{A}$ is a hereditary abelian category. i.e. Δ gives a coalgebra structure on $\mathcal{H}(\mathcal{A})$. Unfortunately, Δ is not a homomorphism of algebras if $\mathcal{A}$ is an exact category. In the rest of this section, let's focus on the case of the exact category $\mathcal{A} = Rep_R^f(\Gamma)$.

The following counterexample shows that $\Delta : \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{A})$, defined in (3), cannot be an algebra homomorphism under any twist in the case of $\mathcal{A} = Rep_R^f(\Gamma)$.

Example 1. Let $\Gamma = A_2 : 1 \to 2$, $R = k[t]/(t^n)$ $(n > 2)$, and $M = N = (R \xrightarrow{t} R)$.

If Δ is an algebra homomorphism, we must have

$$\Delta(MN) = \Delta(M)\Delta(N). \tag{5}$$

On the right hand side of (5), we have

$$\begin{tikzcd}
D \arrow[r, tail] \arrow[d] & X \arrow[r, two heads] & B \arrow[d] \\
M \arrow[d] & & N \arrow[d] \\
C \arrow[r, tail] & Y \arrow[r, two heads] & A,
\end{tikzcd}$$

where the only possible choices for B and D are $0 \xrightarrow{0} R$, $R \xrightarrow{t} R$, and $0 \xrightarrow{0} 0$. Thus, all possible choices for X are $0 \xrightarrow{0} R^2$, $0 \xrightarrow{0} R$, $0 \xrightarrow{0} 0$, $R \xrightarrow{t} R$, $R \xrightarrow{\begin{bmatrix} t \\ 0 \end{bmatrix}} R^2$, and $R^2 \xrightarrow{\begin{bmatrix} t & a \\ 0 & t \end{bmatrix}} R^2$, where $a \in R$.

On the left hand side of (5), we have

$$\begin{tikzcd} & X \arrow[d] & \\ M \arrow[r, tail] & E \arrow[r, two heads] \arrow[d, two heads] & N \\ & Y & \end{tikzcd}$$

Here $E \simeq (R^2 \xrightarrow{\begin{bmatrix} t & a \\ 0 & t \end{bmatrix}} R^2)$. If $a \in tR$, then

$$\begin{bmatrix} t & a \\ 0 & t \end{bmatrix} \simeq \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}.$$

If a is invertible in R, then

$$\begin{bmatrix} t & a \\ 0 & t \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 \\ 0 & t^2 \end{bmatrix}.$$

Let $E_1 \simeq (R^2 \xrightarrow{\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}} R^2)$ and $E_2 \simeq (R^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & t^2 \end{bmatrix}} R^2)$. Then, $MN = \alpha E_1 + \beta E_2$ for some nonzero number α and β. It is clear that $\Delta(E_2)$ has a summand $(R \xrightarrow{1} R) \otimes (R \xrightarrow{t^2} R)$. This term, however, never appears on the right hand side of (5). This shows Δ cannot be an algebra homomorphism.

This counterexample shows that a coalgebra structure of $\mathcal{H}(RA_2)$ cannot be defined by (3) no matter how the multiplication of $\mathcal{H}(RA_2) \otimes \mathcal{H}(RA_2)$ is twisted.

Let $C\mathcal{H}_R(\Gamma)$ be the subalgebra of $\mathcal{H}(R\Gamma)$ generated by all S_i, for $i \in I$. We will give a geometric approach of $C\mathcal{H}_R(\Gamma)$ in the rest of this paper.

4. Lusztig's geometric setting

In this section, we will fix $R = k[t]/(t^n)$, a loop-free quiver $\Gamma = (I, H, s, t)$, and an I-graded free R-module $V = \bigoplus_{i \in I} V_i$ which can be thought of as an I-graded k-vector space. We define

$$E_V^k = \bigoplus_{h \in H} \mathrm{Hom}_k(V_{h'}, V_{h''}), \tag{6}$$

$$E_V^R = \bigoplus_{h \in H} \mathrm{Hom}_R(V_{h'}, V_{h''}), \tag{7}$$

$$G_V^k = \bigoplus_{i\in I} GL_k(V_i), \tag{8}$$

and

$$G_V^R = \bigoplus_{i\in I} GL_R(V_i). \tag{9}$$

G_V^R (resp. G_V^k) acts on E_V^R (resp. E_V^k) by conjugation, i.e., $gx = x'$ and $x'_h = g_{h''} x_h g_{h'}^{-1}$ for all $h \in H$.

Given R-modules V_1 and V_2, $\mathrm{Hom}_k(V_1, V_2)$ has an R-module structure as follows,

$$(rf)(v) = f(rv) - rf(v),$$

for all $r \in R$, $v \in V_1$ and $f \in \mathrm{Hom}_k(V_1, V_2)$. Then

$$\mathrm{Hom}_R(V_1, V_2) = \{f \in \mathrm{Hom}_k(V_1, V_2) \mid rf - fr = 0,\ \forall r \in R\}.$$

Since E_V^k is an affine k-variety and $rf - fr = 0$ for different $r \in R$ are algebraic equations, E_V^R is a closed k-subvariety of E_V^k. Similarly, G_V^R is a closed algebraic k-subgroup of G_V^k.

4.1. **Flags.** A generalized *k-flag* of type $(\underline{i}, \underline{k}) = ((i_1, k_1), \cdots, (i_m, k_m)) \in (I \times \mathbb{N})^m$ in an I-graded k-vector space V is a sequence

$$\mathfrak{f} = (V = V^0 \supset V^1 \supset \cdots \supset V^m = 0)$$

of I-graded vector spaces such that $V^{l-1}/V^l \simeq k^{\oplus k_l}$ concentrated at vertex i_l for all $l = 1, 2, \cdots, m$.

Let $\mathcal{F}^k_{V,\underline{i},\underline{k}}$ be the k-variety of all k-flags of type $(\underline{i}, \underline{k})$ in V.

Let $\widetilde{\mathcal{F}}^k_{V,\underline{i},\underline{k}} = \{(x, \mathfrak{f}) \in E_V^k \times \mathcal{F}^k_{V,\underline{i},\underline{k}} \mid \mathfrak{f} \text{ is } x\text{-stable}\}$, where $\mathfrak{f}$ is x-stable if $x_h(V^l_{h'}) \subset V^l_{h''}$, for all $h \in H, l = 1, \cdots, m$.

G_V^k acts on $\mathcal{F}^k_{V,\underline{i},\underline{k}}$ by $g \cdot \mathfrak{f} \mapsto g\mathfrak{f}$, where

$$g\mathfrak{f} = (gV^0 \supset gV^1 \supset \cdots \supset gV^m = 0)$$

if $\mathfrak{f} = (V = V^0 \supset V^1 \supset \cdots \supset V^m = 0)$. And G_V^k acts diagonally on $\widetilde{\mathcal{F}}^k_{V,\underline{i},\underline{k}}$, i.e., $g \cdot (x, \mathfrak{f}) \mapsto (gx, g\mathfrak{f})$.

4.1.1. An *R-flag* of type $(\underline{i}, \underline{k}) = ((i_1, k_1), \cdots, (i_m, k_m)) \in (I \times \mathbb{N})^m$ in an I-graded R-module V is a sequence

$$\mathfrak{f} = (V = V^0 \supset V^1 \supset \cdots \supset V^m = 0)$$

of I-graded R-modules such that $V^{l-1}/V^l \simeq k^{\oplus k_l}$ as k-vector spaces concentrated at vertex i_l for all $l = 1, 2, \cdots, m$.

Similarly, let $\mathcal{F}^R_{V,\underline{i},\underline{k}}$ be the k-variety of all R-flags of type $(\underline{i}, \underline{k})$ in V.

For any free R-module V, V/tV is a k vector space; we will denote it by V_0. Moreover, we can define the evaluation map as follows,

$$e : \mathcal{F}^R_{V,\underline{i},\underline{k}} \to \mathcal{F}^k_{V_0,\underline{i},\underline{k}}$$
$$\mathfrak{f} = (V \supset V^1 \supset \cdots \supset V^m = 0) \mapsto e(\mathfrak{f}) = (V_0 \supset V_0^1 \supset \cdots \supset V_0^m = 0).$$

Denote

$$\mathfrak{f}_0 = e(\mathfrak{f}) \otimes_k R := (V_0 \otimes_k R \supset V_0^1 \otimes_k R \supset \cdots \supset V_0^m \otimes_k R). \tag{10}$$

Let $\widetilde{\mathcal{F}}^R_{V,\underline{i},\underline{k}} = \{(x,\mathfrak{f}) \in E^k_V \times \mathcal{F}^R_{V,\underline{i},\underline{k}} \mid \mathfrak{f}_0 \text{ is } x\text{-stable}\}$. Moreover, we can define G^R_V actions on $\mathcal{F}^R_{V,\underline{i},\underline{k}}$ and $\widetilde{\mathcal{F}}^R_{V,\underline{i},\underline{k}}$ in a similar way.

Note that if V is an R-module, then a k-subspace $W \subset V$ is an R-submodule if and only if $(1+t)W = W$. This gives an algebraic equation. So $\mathcal{F}^R_{V,\underline{i},\underline{k}}$ (resp. $\widetilde{\mathcal{F}}^R_{V,\underline{i},\underline{k}}$) is a closed subvariety of $\mathcal{F}^k_{V,\underline{i},\underline{k}}$ (resp. $\widetilde{\mathcal{F}}^k_{V,\underline{i},\underline{k}}$).

4.1.2. A *R-free flag* of type $(\underline{i},\underline{k}) = ((i_1,k_1),\cdots,(i_m,k_m)) \in (I\times\mathbb{N})^m$ in an I-graded free R-module V is a sequence

$$\mathfrak{f} = (V = V^0 \supset V^1 \supset \cdots \supset V^m = 0)$$

of I-graded free R-modules such that $V^{l-1}/V^l \simeq R^{\oplus k_l}$ concentrated at vertex i_l as R-modules for all $l = 1,2,\cdots,m$.

Let $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}} \subset \mathcal{F}^R_{V,\underline{i},n\underline{k}}$ be the subvariety of all R-free flags of type $(\underline{i},\underline{k})$, where $(\underline{i},n\underline{k}) = ((i_1,nk_1),\cdots,(i_m,nk_m))$ if $(\underline{i},\underline{k}) = ((i_1,k_1),\cdots,(i_m,k_m))$.

Let $\widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}} = \widetilde{\mathcal{F}}^R_{V,\underline{i},\underline{k}} \cap (E^R_V \times \mathcal{F}^{Rf}_{V,\underline{i},\underline{k}})$ and $\mathcal{F}^{RNf}_{V,\underline{i},\underline{k}}$ (resp. $\widetilde{\mathcal{F}}^{RNf}_{V,\underline{i},\underline{k}}$) be the complement of $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$ (resp. $\widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}}$) in $\mathcal{F}^R_{V,\underline{i},n\underline{k}}$ (resp. $\widetilde{\mathcal{F}}^R_{V,\underline{i},n\underline{k}}$). We can define G^R_V actions on these k-varieties in a similar way.

Remark 1. Notice that $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$ is an $(n-1)$th-Jet scheme over $\mathcal{F}^k_{V_0,\underline{i},\underline{k}}$ (see [25]). So $\dim_k \mathcal{F}^{Rf}_{V,\underline{i},\underline{k}} = n \dim_k \mathcal{F}^k_{V_0,\underline{i},\underline{k}}$. Moreover, the dimension (resp. shift functors for perverse sheaves) argument in Lusztig's papers can be adapted here by multiplying by n. In the rest of this section, we will skip the proof of the statements about dimension and shift degree.

To simplify the notations, for any $(\underline{i},\underline{k}) \in (I\times\mathbb{N})^m$ and each $i \in I$, let $N_i(\underline{i},\underline{k}) = \sum_{r<r'} k_r k_{r'} \delta_{ii_r}\delta_{ii_{r'}}$; for each $h \in H$, let $N_h(\underline{i},\ \underline{k}) = \sum_{r'<r} k_{r'}k_r \delta_{h'i_{r'}}\delta_{h''i_r}$, where δ is the Kronecker delta. In the following, dimension always refers to k-dimension, so we will denote it by dim instead of $\dim_k$. Rank always refers to the rank of free R-modules, and we will therefore denote it by Rank.

Proposition 1. (1) *$\mathcal{F}^R_{V,\underline{i},\underline{k}}$ is a projective variety.*

(2) *$\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$ is an open subvariety of $\mathcal{F}^R_{V,\underline{i},n\underline{k}}$ and $\mathcal{F}^{RNf}_{V,\underline{i},\underline{k}}$ is a closed subvariety of $\mathcal{F}^R_{V,\underline{i},n\underline{k}}$.*

(3) *The evaluation map $e : \mathcal{F}^{Rf}_{V,\underline{i},\underline{k}} \to \mathcal{F}^k_{V_0,\underline{i},\underline{k}}$ is a vector bundle with rank $(n-1)\sum_i N_i(\underline{i},\underline{k})$. Hence the dimension of $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$ is $n\sum_i N_i(\underline{i},\underline{k})$.*

Proof. (1) Choose $\mathfrak{f} \in \mathcal{F}^k_{V,\underline{i},\underline{k}}$. Define $P := \mathrm{Stab}_{G^k_V}(\mathfrak{f})$, the stabilizer of $\mathfrak{f}$ in G^k_V. P is a parabolic subgroup, so $\mathcal{F}^k_{V,\underline{i},\underline{k}} = G^k_V/P$ is a projective variety. $\mathcal{F}^R_{V,\underline{i},\underline{k}}$ is a closed subvariety, so it is also a projective variety.

(2) To show $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}} \subset \mathcal{F}^R_{V,\underline{i},n\underline{k}}$ is an open subset, let's first consider the Grassmannian.

Let $G_k(sn,ln)$ be the set of all sn-dimensional k-subspaces in V with $\dim V = ln$ and $G_R(sn,ln) = \{\mathfrak{f} \in G_k(sn,ln) \mid (1+t)\mathfrak{f} = \mathfrak{f}\}$. Let $G_{Rf}(s,l)$ be the set of all free R-submodules with Rank s in V, where $\mathrm{Rank}(V) = l$. Clearly, $G_{Rf}(s,l) \subset G_R(sn,ln) \subset G_k(sn,ln)$.

Let $\widetilde{G}_R(sn,ln) = \{(W,b_W) \mid W \in G_R(sn,ln) \text{ and } b_W \text{ is a } k\text{-basis of } W\}$.

The first projection $\pi : \widetilde{G}_R(sn,ln) \to G_R(sn,ln)$ is a frame bundle.

Define

$$\phi : \widetilde{G}_R(sn, ln) \to \mathrm{Mat}(sn),$$
$$(W, b_W) \mapsto M(b_W, t),$$

where $\mathrm{Mat}(sn)$ is the set of all $sn \times sn$ matrices and $M(b_W, t)$ is the matrix of t under the basis b_W. Clearly, ϕ is a morphism of algebraic varieties.

In general, for any free R-module $V \simeq R^{\oplus r}$, the R-module structure induces a nilpotent k-linear map $t : V \to V$ where $\dim(\mathrm{Ker}(t)) = r$.

For any R-submodule $W \subset V$ with k-dimension ns, W is a free R-module if and only if $\dim(\mathrm{Ker}(t|_W)) = s$, i.e., $t|_W$ has maximal rank $(n-1)s$. Therefore, $G_{Rf}(s, l) = \pi(\phi^{-1}(\mathrm{Mat}(sn)_{rk=s(n-1)}))$, where $\mathrm{Mat}(sn)_{rk=s(n-1)}$ is the set of all matrices with rank $s(n-1)$. $\phi^{-1}(\mathrm{Mat}(sn)_{rk=s(n-1)})$ is open in $\phi^{-1}(\mathrm{Mat}(sn)_{rk\leq s(n-1)}) = \widetilde{G}_R(sn, ln)$ since $\mathrm{Mat}(sn)_{rk=s(n-1)}$ is an open subset in $\mathrm{Mat}(sn)_{rk\leq s(n-1)}$. Moreover, π is a principle $GL_{sn}(k)$-bundle and $\phi^{-1}(\mathrm{Mat}(sn)_{rk=s(n-1)})$ is $GL_{sn}(k)$-stable, $G_{Rf}(s, l)$ is open in $G_R(sn, ln)$.

Now for any flag $(V^0 \supset V^1 \supset \cdots \supset V^m) \in \mathcal{F}^R_{V,\underline{i},n\underline{k}}$, each entry V^l gives an open condition when it is a free R-module. So $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$ is the intersection of m many such open subsets. The smoothness follows from Remark 1 and the notes after Lemma 1.2 in [25]. This proves the first statement. The second statement follows from the first one.

(3) Recall for any R-free module V, we denote $V_0 = V/tV$, then $V \simeq V_0 \otimes_k R$. Without loss of generality, we will simply assume $V = V_0 \otimes_k R$. For any element $\mathfrak{f} = (V_0 \supset V_0^1 \supset \cdots \supset V_0^m) \in \mathcal{F}^k_{V_0,\underline{i},\underline{k}}$, let $P^k(\mathfrak{f})$ be its stabilizer in $G^k_{V_0}$. Let $P^R(\mathfrak{f})$ be the stabilizer of $\mathfrak{f} \otimes_k R$ (see (10)) in G^R_V, where we identify V with $V_0 \otimes_k R$. So $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$ (resp. $\mathcal{F}^k_{V_0,\underline{i},\underline{k}}$) can be identified with $G^R_V/P^R(\mathfrak{f})$ (resp. $G^k_{V_0}/P^k(\mathfrak{f})$) for a fixed $\mathfrak{f}$.

Now consider the map $\iota : \mathcal{F}^k_{V_0,\underline{i},\underline{k}} \to \mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$, $\mathfrak{f} \mapsto \mathfrak{f} \otimes_k R$. Since ι is an injective map, in the rest of this section, we will identify $\mathcal{F}^k_{V_0,\underline{i},\underline{k}}$ with $\iota(\mathcal{F}^k_{V_0,\underline{i},\underline{k}})$ and consider $\mathcal{F}^k_{V_0,\underline{i},\underline{k}}$ as a subset of $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$. Similarly, we will consider $G^k_{V_0}/P^k(\mathfrak{f})$ as a subset of $G^R_V/P^R(\mathfrak{f})$ via the map ι.

Now any element $A \in G^R_V$ can be written as $A = h \cdot A_0$, where $A_0 \in G^k_{V_0}$ and $h \in H := \{Id + tB | B \in \mathrm{End}_R(V)\}$. Then the evaluation map e sends $h \cdot A_0$ to A_0. Therefore, $\forall x \in G^k_{V_0}$, we have $e^{-1}(x) = H/(H \cap P^R) \cdot x$. As a set, $H/(H \cap P^R) \cdot x$ is in 1-1 correspondence to $H/(H \cap P^R)$, and $H/(H \cap P^R)$ is a direct sum of quasi-lower triangular matrices with entries in tR for all $i \in I$, which is clearly a k-vector space of dimension $(n-1)\sum_i N_i(\underline{i}, \underline{k})$.

Now for any open subset $U \subset G^k_{V_0}/P^k(\mathfrak{f})$, define

$$\phi_U : U \times H/(H \cap P^R) \to e^{-1}(U)$$
$$(x, a) \mapsto a \cdot x.$$

It is easy to check this gives a vector bundle structure. □

Remark 2. From the above proof, we have $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}} = \left\{ gh \cdot \mathfrak{f} \mid g \in G^k_{V_0}/P^k, h \in H/(H \bigcap P^R) \right\}$ for a fixed $\mathfrak{f} \in \mathcal{F}^k_{V_0,\underline{i},\underline{k}}$ and $e : \mathcal{F}^{Rf}_{V,\underline{i},\underline{k}} \to \mathcal{F}^k_{V_0,\underline{i},\underline{k}}$ sending $gh \cdot \mathfrak{f}$ to $g \cdot \mathfrak{f}$. Since $G^k_{V_0}$ acts on $\mathcal{F}^k_{V_0,\underline{i},\underline{k}}$ transitively, we have $\mathcal{F}^k_{V_0,\underline{i},\underline{k}} = \left\{ g \cdot \mathfrak{f} \mid g \in G^k_{V_0}/P^k(\mathfrak{f}) \right\}$ for a fixed $\mathfrak{f} \in \mathcal{F}^k_{V_0,\underline{i},\underline{k}}$.

Proposition 2. (1) *$\widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}}$ is a smooth irreducible variety, and the second projection $p_2 : \widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}} \to \mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$ is a vector bundle of dimension $n\sum_h N_h(\underline{i},\underline{k})$. So the k-dimension of $\widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}}$ is $d(V,\underline{i},\underline{k}) := n\sum_i N_i(\underline{i},\underline{k}) + n\sum_h N_h(\underline{i},\underline{k})$.*

(2) *Let $\pi^f_{V,\underline{i},\underline{k}} : \widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}} \to E^R_V$ be the first projection map. Then $(\pi^f_{V,\underline{i},\underline{k}})_!\mathbf{1}$ is semisimple.*

Proof. Let $\widetilde{\mathcal{F}}^k_R = \Big\{(x,\mathfrak{f}) \in E^R_V \times \mathcal{F}^k_{V_0,\underline{i},\underline{k}} \mid \mathfrak{f} \text{ is } x-\text{stable}\Big\}$. Here we consider $\mathcal{F}^k_{V_0,\underline{i},\underline{k}}$ as a subset of $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$. By using Lusztig's argument for Lemma 1.6 in [18], we want to show the second projection $p_2 : \widetilde{\mathcal{F}}^k_R \to \mathcal{F}^k_{V_0,\underline{i},\underline{k}}$ is a vector bundle. In fact, for any $\mathfrak{f} = (V \supset V^1 \supset \cdots \supset V^m = 0) \in \mathcal{F}^k_{V_0,\underline{i},\underline{k}}$, let Z be the fiber of p_2. The first projection identifies Z with the set of all $x \in E^R_V$ such that $x_h(V^l_{h'}) \subset V^l_{h''}$ for all $h \in H$ and all $l = 0,1,\cdots,m$. This is a linear subspace of E^R_V because we can choose a basis for each V_i such that x_h are upper triangle matrices for each $h \in H$. Hence its dimension is equal to

$$n\sum_{l'\le l, h\in H}(\mathrm{Rank}(V^{l'-1}_{h'}) - \mathrm{Rank}(V^{l'}_{h'}))(\mathrm{Rank}(V^{l'-1}_{h''}) - \mathrm{Rank}(V^{l'}_{h''}))$$

which is equal to $n\sum_h N_h(\underline{i},\underline{k})$. Since G^R_V acts on $\mathcal{F}^k_{V_0,\underline{i},\underline{k}}$ transitively, this is independent of $\mathfrak{f}$ but only dependent on $(\underline{i},\underline{k})$. This shows that p_2 is a vector bundle.

Now consider the following cartesian square,

(11)
$$\begin{tikzcd}
\widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}} \arrow[r, "p_1"] \arrow[d, "b"] & \mathcal{F}^{Rf}_{V,\underline{i},\underline{k}} \arrow[d, "e"] \\
\widetilde{\mathcal{F}}^k_R \arrow[r, "p_2"] & \mathcal{F}^k_{V_0,\underline{i},\underline{k}}.
\end{tikzcd}$$

Since p_2 is a vector bundle with rank $n\sum_h N_h(\underline{i},\underline{k})$, p_1 is a vector bundle with rank $n\sum_h N_h(\underline{i},\underline{k})$. The smoothness and irreducibility follow Proposition 1. This proves the first statement. The second statement follows from the first one.

(2) Consider cartesian square (11). By Proposition 1, e is a vector bundle, then b is also a vector bundle. Now consider the following commutative diagram

(12)
$$\begin{tikzcd}
\widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}} \arrow[r, "b"] \arrow[dr, "\pi^f_{V,\underline{i},\underline{k}}"'] & \widetilde{\mathcal{F}}^k_R \arrow[d, "p"] \\
 & E^R_V
\end{tikzcd}$$

Here the first projection map, p, is a proper map. By Proposition 2.2.3(3), $(\pi^f_{V,\underline{i},\underline{k}})_!\mathbf{1} = p_!b_!\mathbf{1}$ is semisimple. Proposition follows. ☐

Denote $\widetilde{L}^f_{V,\underline{i},\underline{k}} := (\pi^f_{V,\underline{i},\underline{k}})_!\mathbf{1} \in \mathcal{D}(E_V)$. By Proposition 2, $\widetilde{L}^f_{V,\underline{i},\underline{k}}$ is semisimple.

Proposition 3. *Let $L^f_{V,\underline{i},\underline{k}} = \widetilde{L}^f_{V,\underline{i},\underline{k}}[d(V,\underline{i},\underline{k}) + (n-1)\sum_i N_i(\underline{i},\underline{k})]$. Then $L^f_{V,\underline{i},\underline{k}}$ is a perverse sheaf. In particular, $\mathbb{D}(L^f_{V,\underline{i},\underline{k}}) = L^f_{V,\underline{i},\underline{k}}$.*

Proof. From the proof of Proposition 2, we have $\pi^f_{V,\underline{i},\underline{k}} = pb$, where p is a proper map and b is a vector bundle with rank $(n-1)\sum_i N_i(\underline{i},\underline{k})$. Therefore,

$$\begin{aligned}
L^f_{V,\underline{i},\underline{k}} &= (\pi^f_{V,\underline{i},\underline{k}})_! \mathbf{1}[d(V,\underline{i},\underline{k}) + (n-1)\sum_i N_i(\underline{i},\underline{k})] \\
&= p_! b_! \mathbf{1}[d(V,\underline{i},\underline{k}) + (n-1)\sum_i N_i(\underline{i},\underline{k})] \\
&= p_! b_! b^* \mathbf{1}_{\widetilde{\mathcal{F}}^k_R}[d(V,\underline{i},\underline{k}) + (n-1)\sum_i N_i(\underline{i},\underline{k})] \\
&= p_! \mathbf{1}_{\widetilde{\mathcal{F}}^k_R}[d(V,\underline{i},\underline{k}) - (n-1)\sum_i N_i(\underline{i},\underline{k})] \\
&= p_! \mathbf{1}_{\widetilde{\mathcal{F}}^k_R}[\dim_k(\widetilde{\mathcal{F}}^k_R)].
\end{aligned}$$

Since p is a proper map and $\mathbf{1}_{\widetilde{\mathcal{F}}^k_R}[\dim_k(\widetilde{\mathcal{F}}^k_R)]$ is a perverse sheaf, $L^f_{V,\underline{i},\underline{k}}$ is a perverse sheaf. $\square$

Similarly, let $\pi^R_{V,\underline{i},\underline{k}} : \widetilde{\mathcal{F}}^R_{V,\underline{i},\underline{k}} \to E^R_V$ be the first projection. We define $\widetilde{L}^R_{V,\underline{i},\underline{k}} = (\pi^R_{V,\underline{i},\underline{k}})_! \mathbf{1}_{\widetilde{\mathcal{F}}^R_{V,\underline{i},\underline{k}}}$.

Let $\mathcal{P}^f_V$ (resp. $\mathcal{P}^R_V$) be the full subcategory of $\mathcal{M}(E^R_V)$ consisting of perverse sheaves which are direct sums of the simple perverse sheaves L that are the direct summands of $\widetilde{L}^f_{V,\underline{i},\underline{k}}$ (resp. $\widetilde{L}^R_{V,\underline{i},\underline{k}}$) up to shift for some $(\underline{i},\underline{k}) \in (I\times\mathbb{N})^m$. Let $\mathcal{Q}^f_V$ (resp. $\mathcal{Q}^R_V$) be the full subcategory of $\mathcal{D}(E^R_V)$ whose objects are isomorphic to finite direct sums of $L[d]$ for various simple perverse sheaves $L \in \mathcal{P}^f_V$ (resp. $\mathcal{P}^R_V$) and various $d \in \mathbb{Z}$.

4.2. **Restriction functor.** To define the restriction functor, Lusztig considers the following diagram

$$E_T \times E_W \xleftarrow{\kappa} F \xrightarrow{\iota} E_V,$$

where ι is an embedding and $\kappa(x) = (x_W, x_T)$. Recall $x_W = x|_W$ and x_T is the induced map $\overline{x} : V/W \to V/W$. For any $B \in \mathcal{D}(E^R_V)$, define $\overline{\mathrm{Res}}^V_{T,W} B := \kappa_! \iota^* B$. However, it is no longer true that $\overline{\mathrm{Res}}^V_{T,W} B \in \mathcal{Q}^f_{T,W}$, even for $B \in \mathcal{Q}^f_V$. In fact, given a free flag $\mathfrak{f} = (V^0 \supset V^1 \supset \cdots \supset V^m = 0)$, and $W \subset V$, $T = V/W$ being free R-modules, the induced flags

$$\mathfrak{f}_T := ((V^0+W)/W \supset (V^1+W)/W \supset \cdots \supset (V^m+W)/W = 0) \tag{13}$$

$$\mathfrak{f}_W := (V^0 \bigcap W \supset V^1 \bigcap W \cdots \supset V^m \bigcap W = 0) \tag{14}$$

are no longer free flags, since $V^l \bigcap W$ and $(V^l+W)/W$ are no longer free modules in general.

Lemma 2. *$\overline{\mathrm{Res}}^V_{T,W}(B)$ is semisimple in $\mathcal{D}^b_{G^R_T\times G^R_W}(E^R_T \times E^R_W)$ for $B \in \mathcal{Q}^f_V$.*

Proof. It is sufficient to prove that $\kappa_!\iota^*(\widetilde{L}^f_{V,\underline{i},\underline{k}})$ is semisimple, that is, $k_!\iota^*(\pi^f_{V,\underline{i},\underline{k}})_!(\mathbf{1}_{\widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}}})$ is semisimple. Consider the following diagram,

$$\begin{tikzcd} & & \widetilde{F}^R \arrow[r, "\iota'"] \arrow[dll, "\kappa\pi'"'] \arrow[d, "\pi'"'] & \widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}} \arrow[d, "\pi^f_{V,\underline{i},\underline{k}}"] \\ E_T \times E_W & & F \arrow[ll, "\kappa"'] \arrow[r, "\iota"] & E_V, \end{tikzcd} \tag{15}$$

where $\widetilde{F}^R = (\pi^f_{V,\underline{i},\underline{k}})^{-1}(F)$.

Using base change, we have

$$\kappa_!\iota^*\pi_{V,\underline{i},\underline{k}!}\mathbf{1} = \kappa_!\pi'_!\iota'^*\mathbf{1} = (\kappa\pi')_!\mathbf{1}_{\widetilde{F}^R}.$$

We now prove that $(\kappa\pi')_!\mathbf{1}_{\widetilde{F}^R}$ is semisimple.

Recall from the proof of Proposition 2(2), we have a vector bundle $\widetilde{F}^R \xrightarrow{b} \widetilde{\mathcal{F}}^k_R$, where $\widetilde{\mathcal{F}}^k_R = \Big\{(x,\mathfrak{f}) \in F \times \mathcal{F}^k_{V_0,\underline{i},\underline{k}} \mid \mathfrak{f} \text{ is } x-\text{stable}\Big\}$.

For any $\underline{k}_1$ and $\underline{k}_2$ satisfying $\underline{k} = \underline{k}_1 + \underline{k}_2$, we set

$$\widetilde{\mathcal{F}}_{R,\underline{i},\underline{k}_1,\underline{k}_2} = \Big\{(x,\mathfrak{f}) \in \widetilde{\mathcal{F}}^k_R \mid (x_T,\mathfrak{f}_T) \in \widetilde{\mathcal{F}}^k_{T_0,\underline{i},\underline{k}_1},\ (x_W,\mathfrak{f}_W) \in \widetilde{\mathcal{F}}^k_{W_0,\underline{i},\underline{k}_2}\Big\},$$

where $\mathfrak{f}_T$ and $\mathfrak{f}_W$ are defined in (13), (14). $\widetilde{\mathcal{F}}_{R,\underline{i},\underline{k}_1,\underline{k}_2}$ is a locally closed subvariety of $\widetilde{\mathcal{F}}^k_R$. For various $(\underline{i},\underline{k}_1)$ and $(\underline{i},\underline{k}_2)$, $\widetilde{\mathcal{F}}_{R,\underline{i},\underline{k}_1,\underline{k}_2}$ form a partition of $\widetilde{\mathcal{F}}^k_R$. Then $b^{-1}(\widetilde{\mathcal{F}}_{R,\underline{i},\underline{k}_1,\underline{k}_2})$ for various $(\underline{i},\underline{k}_1)$ and $(\underline{i},\underline{k}_2)$ form a partition of $\widetilde{\mathcal{F}}^R$.

Now define $\alpha_{\underline{i},\underline{k}_1,\underline{k}_2} : \widetilde{\mathcal{F}}_{R,\underline{i},\underline{k}_1,\underline{k}_2} \to \widetilde{\mathcal{F}}^k_{T_0,\underline{i},\underline{k}_1} \times \widetilde{\mathcal{F}}^k_{W_0,\underline{i},\underline{k}_2}$, which sends $(x,\mathfrak{f})$ to $((x_T,\mathfrak{f}_T),(x_W,\mathfrak{f}_W))$. It is easy to check that $\alpha_{\underline{i},\underline{k}_1,\underline{k}_2}$ is a vector bundle.

Let $D_j = \Big\{(\underline{i},\underline{k}_1,\underline{k}_2) \mid \dim_k(\widetilde{\mathcal{F}}_{R,\underline{i},\underline{k}_1,\underline{k}_2}) = j\Big\}$. Let $\widetilde{\mathcal{F}}^k_{Rj}$ be the disjoint union of $\widetilde{\mathcal{F}}_{R,\underline{i},\underline{k}_1,\underline{k}_2}$ for various $(\underline{i},\underline{k}_1,\underline{k}_2) \in D_j$. i.e. $\widetilde{\mathcal{F}}^k_{Rj} = \coprod_{(\underline{i},\underline{k}_1,\underline{k}_2)\in D_j} \widetilde{\mathcal{F}}_{R,\underline{i},\underline{k}_1,\underline{k}_2}$. Let $Z_j = \coprod_{(\underline{i},\underline{k}_1,\underline{k}_2)\in D_j}(\widetilde{\mathcal{F}}^k_{T_0,\underline{i},\underline{k}_1} \times \widetilde{\mathcal{F}}^k_{W_0,\underline{i},\underline{k}_2})$. There is a well-defined map $\alpha_j := \coprod_{(\underline{i},\underline{k}_1,\underline{k}_2)\in D_j} \alpha_{\underline{i},\underline{k}_1,\underline{k}_2} : \widetilde{\mathcal{F}}^k_{Rj} \to Z_j$. Since these are disjoint union, α_j is a vector bundle. Moreover, the composition map $\alpha_j \circ b : b^{-1}(\widetilde{\mathcal{F}}^k_{Rj}) \to Z_j$ is a vector bundle. Therefore, we have the following diagram,

$$b^{-1}(\widetilde{\mathcal{F}}^k_{Rj}) \xrightarrow{\alpha_j \circ b} Z_j \xrightarrow{\pi_j} E_T \times E_W,$$

where $\pi_j := \coprod_{(\underline{i},\underline{k}_1,\underline{k}_2)\in D_j}(\pi_{T,\underline{i}_1,\underline{k}_1} \times \pi_{W,\underline{i}_2,\underline{k}_2})$ is a proper map.

By Proposition 2.2.3(3), $(\kappa\pi')_!\mathbf{1}_{\widetilde{F}^R}$ is semisimple. □

Since the objects in $\mathcal{Q}^R_V$ are semisimple complexes, every object $A \in \mathcal{Q}^R_V$ can be uniquely written into $A = A^f \oplus A^{Nf}$ such that $A^f \in \mathcal{Q}^f_V$, $A^{Nf} \in \mathcal{Q}^R_V \setminus \mathcal{Q}^f_V$ and A^f is the maximal subobject of A which is in $\mathcal{Q}^f_V$. Therefore we can define a projection map $P_f : \mathcal{Q}^R_V \to \mathcal{Q}^f_V$ sending A to A^f.

Definition 2. $\widetilde{\mathrm{Res}}^V_{T,W}(B) := P_f(\overline{\mathrm{Res}}^V_{T,W}(B))$.

Proposition 4. $\widetilde{\mathrm{Res}}^V_{T,W}(B) \in \mathcal{Q}^f_{T,W}$ *if* $B \in \mathcal{Q}_V$.

Proof. This follows directly from the definition of $\widetilde{\mathrm{Res}}^V_{T,W}$. $\square$

Proposition 5. *If $E_T = 0$, i.e., $E_W \simeq F$, then $\overline{\mathrm{Res}}^V_{T,W}(B) = \widetilde{\mathrm{Res}}^V_{T,W}(B)$ for all $B \in \mathcal{Q}^f_V$.*

Proof. Since any simple object $B \in \mathcal{Q}^f_V$ is a direct summand of $\widetilde{L}^f_{V,\underline{i},\underline{k}}$ for some $(\underline{i},\underline{k})$ up to shift, it is enough to prove the proposition for $B = \widetilde{L}^f_{V,\underline{i},\underline{k}}$.

From the proof of Lemma 2 and Diagram 15, if $E_W \simeq F$, κ is an isomorphism, then

$$\overline{\mathrm{Res}}^V_{T,W}(\widetilde{L}^f_{V,\underline{i},\underline{k}}) = \kappa_! \iota^* (\pi^f_{V,\underline{i},\underline{k}})_! \mathbf{1} = (\kappa\pi')_! \iota'^* \mathbf{1} = (\kappa\pi')_! \mathbf{1}_{\widetilde{F}^R} \in \mathcal{Q}^f_{T,W}.$$

Here $\mathcal{Q}^f_{T,W}$ is defined similarly as $\mathcal{Q}^f_V$ for $E_T \times E_W$.

So $\overline{\mathrm{Res}}^V_{T,W}(\widetilde{L}^f_{V,\underline{i},\underline{k}}) = \widetilde{\mathrm{Res}}^V_{T,W}(\widetilde{L}^f_{V,\underline{i},\underline{k}})$ by the definition of $\widetilde{\mathrm{Res}}^V_{T,W}$. $\square$

4.3. **Induction functor.** By abuse of notation, in the rest of this section, we will write E_V (resp. G_V) instead of E^R_V (resp. G^R_V) unless we specify. And $\widetilde{\mathcal{F}}_{V,\underline{i},\underline{k}}$ always means $\widetilde{\mathcal{F}}^f_{V,\underline{i},\underline{k}}$ unless we specify.

Let W be an I-graded free R-submodule of V such that $T = V/W$ is also a free R-module. Let P be the stabilizer of W in G_V and U be the unipotent radical of P. Consider the following diagram:

$$E_T \times E_W \xleftarrow{p_1} G_V \times^U F \xrightarrow{p_2} G_V \times^P F \xrightarrow{p_3} E_V. \tag{16}$$

Here $p_1(g,x) = \kappa(x)$, $p_2(g,x) = (g,x)$, and $p_3(g,x) = g(\iota(x))$, where κ and ι are the maps introduced in Section 4.2. For any $A \in \mathcal{D}_{G_T \times G_W}(E_T \times E_W)$, define $\widetilde{\mathrm{Ind}}^V_{T,W} A := p_{3!} p_{2\flat} p_1^* A$. Here $p_{2\flat}$ is well defined since p_2 is a principle $G_T \times G_W$-bundle.

Proposition 6. $\widetilde{\mathrm{Ind}}^V_{T,W} A \in \mathcal{Q}^f_V$ *if* $A \in \mathcal{Q}^f_{T,W}$.

Proof. Since $\widetilde{\mathrm{Ind}}^V_{T,W}$ is additive, it is enough to prove the proposition for $A = \widetilde{L}_{T,\underline{i}',\underline{k}'} \boxtimes \widetilde{L}_{W,\underline{i}'',\underline{k}''}$, where $(\underline{i}',\underline{k}') = ((i_1,k_1),\cdots,(i_m,k_m))$ and $(\underline{i}'',\underline{k}'') = ((i_{m+1},k_{m+1}),\cdots,(i_{m+s},k_{m+s}))$. Let $(\underline{i},\underline{k}) = ((\underline{i}',\underline{k}'),(\underline{i}'',\underline{k}'')) := ((i_1,k_1),\cdots,(i_m,k_m),(i_{m+1},k_{m+1}),\cdots,(i_{m+s},k_{m+s}))$.

Let $\mathcal{F}^0_{V,\underline{i},\underline{k}} = \{(V^0 \supset V^1 \supset \cdots \supset V^m \cdots \supset V^{m+s} = 0) \in \mathcal{F}_{V,\underline{i},\underline{k}} \mid V^m = W\}$ and $\widetilde{\mathcal{F}}^0_{V,\underline{i},\underline{k}} = \widetilde{\mathcal{F}}_{V,\underline{i},\underline{k}} \bigcap (F \times \mathcal{F}^0_{V,\underline{i},\underline{k}})$.

Now consider the following diagram,

$$\begin{tikzcd}
\widetilde{\mathcal{F}}_{T,\underline{i}',\underline{k}'} \times \widetilde{\mathcal{F}}_{W,\underline{i}'',\underline{k}''} \arrow[d, "\pi_{T,W}"] \arrow[dr, phantom, "\boxed{1}"] & G_V \times^U \widetilde{\mathcal{F}}^0_{V,\underline{i},\underline{k}} \arrow[l, "\widetilde{p_1}"'] \arrow[r, "\widetilde{p_2}"] \arrow[d, "u'"] \arrow[dr, phantom, "\boxed{2}"] & G_V \times^P \widetilde{\mathcal{F}}^0_{V,\underline{i},\underline{k}} \arrow[r, "i"] \arrow[d, "u"] & \widetilde{\mathcal{F}}_{V,\underline{i},\underline{k}} \arrow[d, "\pi_{V,\underline{i},\underline{k}}"] \\
E_T \times E_W & G_V \times^U F \arrow[l, "p_1"'] \arrow[r, "p_2"] & G_V \times^P F \arrow[r, "p_3"] & E_V.
\end{tikzcd} \tag{17}$$

Here the vertical maps are all projection maps and i is an identity map. The squares $\boxed{1}$ and $\boxed{2}$ are both cartesian squares and $\widetilde{p_2}$ is a principle $G_T \times G_W$-bundle. It follows that

$$p_1^*(\pi_{T,W})_! \mathbf{1} = u'_! \widetilde{p_1}^* \mathbf{1} = u'_! \widetilde{p_2}^* \mathbf{1} = p_2^* u_! \mathbf{1}.$$

So

$$p_{3!} p_{2\flat} p_1^* A = p_{3!} p_{2\flat} p_1^* (\pi_{T,W})_! \mathbf{1} = p_{3!} u_! \mathbf{1} = (\pi_{V,\underline{i},\underline{k}})_! \mathbf{1} \in \mathcal{Q}^f_V.$$

$\square$

Remark 3. The above proof also shows $\widetilde{\mathrm{Ind}}^V_{T,W}(\widetilde{L}_{T,\underline{i}',\underline{k}'} \boxtimes \widetilde{L}_{W,\underline{i}'',\underline{k}''}) = \widetilde{L}_{V,\underline{i},\underline{k}}$, where $(\underline{i},\underline{k}) = ((\underline{i}',\underline{k}'),(\underline{i}'',\underline{k}''))$.

Lemma 3. *Let $b : Y \to X$ be a fiber bundle with d dimensional connected smooth irreducible fiber. If $B = b^*A$ for some $A \in \mathcal{D}^b(X)$, then $\mathbb{D}b_!B = (b_!\mathbb{D}B)[2d]$.*

Proof. Since $b_!B = b_!b^*A = A[-2d]$, we have

$$\mathbb{D}b_!B = \mathbb{D}(A[-2d]) = (\mathbb{D}A)[2d],$$

and

$$b_!\mathbb{D}B = b_!\mathbb{D}b^*A = b_!b^!(\mathbb{D}A) = b_!b^*(\mathbb{D}A)[2d] = \mathbb{D}A.$$

$\square$

Denote d_1 (resp. d_2) the dimension of the fibers of p_1 (resp. p_2), where p_1 and p_2 are the maps defined in Diagram (16). After simple calculations, $d_2 = \mathrm{dim}P/U$ and $d_1 = \mathrm{dim}G_V/U + n\sum_{h\in H}\mathrm{Rank}(T_{h'})\mathrm{Rank}(W_{h''})$.

Proposition 7. *Let A be a direct summand of $\widetilde{L}_{T,\underline{i},\underline{k}} \boxtimes \widetilde{L}_{W,\underline{j},\underline{l}}$, then*

$$\mathbb{D}(\widetilde{\mathrm{Ind}}^V_{T,W}(A)) = \widetilde{\mathrm{Ind}}^V_{T,W}(\mathbb{D}(A))[2d_1 - 2d_2 + 2(n-1)\sum_i \mathrm{Rank}(T_i)\mathrm{Rank}(W_i)].$$

Proof. Since $\mathbb{D}$ is additive, it is enough to consider $A = \widetilde{L}_{T,\underline{i},\underline{k}} \boxtimes \widetilde{L}_{W,\underline{j},\underline{l}}$. From the proof of Proposition 6, we have

$$\mathbb{D}(\widetilde{\mathrm{Ind}}^V_{T,W}(A)) = \mathbb{D}(p_{3!}p_{2\flat}p_1^*(\pi_{T,W})_!\mathbf{1}) = \mathbb{D}((\pi_{V,(\underline{i},\underline{k}),(\underline{j},\underline{l})})_!i_!\widetilde{p}_{2\flat}\widetilde{p}_1^*\mathbf{1}).$$

From the proof of Proposition 2, $\pi_{V,(\underline{i},\underline{k}),(\underline{j},\underline{l})} = p \circ b$ such that p is a proper map and b is a vector bundle with rank $(n-1)\sum_i N_i((\underline{i},\underline{k}),(\underline{j},\underline{l}))$. By Lemma 3,

$$\begin{aligned}
\mathbb{D}((\pi_{V,(\underline{i},\underline{k}),(\underline{j},\underline{l})})_!i_!\widetilde{p}_{2\flat}\widetilde{p}_1^*\mathbf{1}) &= p_!b_!i_!\mathbb{D}\widetilde{p}_{2\flat}\widetilde{p}_1^*\mathbf{1}[2(n-1)\sum_i N_i((\underline{i},\underline{k}),(\underline{j},\underline{l}))] \\
&= (\pi_{V,(\underline{i},\underline{k}),(\underline{j},\underline{l})})_!i_!\widetilde{p}_{2\flat}\widetilde{p}_1^*(\mathbb{D}\mathbf{1})[2d_1 - 2d_2 + 2(n-1)\sum_i N_i((\underline{i},\underline{k}),(\underline{j},\underline{l}))].
\end{aligned}$$

On the other hand, by the similar reason,

$$\begin{aligned}
\widetilde{\mathrm{Ind}}^V_{T,W}(\mathbb{D}A) &= p_{3!}p_{2\flat}p_1^*(\mathbb{D}(\pi_{T,W})_!\mathbf{1}) \\
&= p_{3!}p_{2\flat}p_1^*p'_!b'_!(\mathbb{D}\mathbf{1})[2(n-1)\sum_i(N_i(\underline{i},\underline{k}) + N_i(\underline{j},\underline{l}))] \\
&= p_{3!}p_{2\flat}p_1^*(\pi_{T,W})_!(\mathbb{D}\mathbf{1})[2(n-1)\sum_i(N_i(\underline{i},\underline{k}) + N_i(\underline{j},\underline{l}))] \\
&= (\pi_{V,(\underline{i},\underline{k}),(\underline{j},\underline{l})})_!i_!\widetilde{p}_{2\flat}\widetilde{p}_1^*(\mathbb{D}\mathbf{1})[2(n-1)\sum_i(N_i(\underline{i},\underline{k}) + N_i(\underline{j},\underline{l}))].
\end{aligned}$$

Here we use a similar decomposition $\pi_{T,W} = p' \circ b'$ such that p' is a proper map and b' is a vector bundle with rank $(n-1)\sum_i(N_i(\underline{i},\underline{k}) + N_i(\underline{j},\underline{l}))$.

By the definition of $N_i(\underline{i},\underline{k})$, it is easy to check that

$$N_i((\underline{i},\underline{k}),(\underline{j},\underline{l}))-N_i(\underline{i},\underline{k})-N_i(\underline{j},\underline{l})=\sum_{r,r'}k_rl_{r'}\delta_{ii_r}\delta_{ij_{r'}}=\mathrm{Rank}(T_i)\mathrm{Rank}(W_i). \tag{18}$$

The proposition follows. $\square$

Let

$$\mathrm{Ind}_{T,W}^V A=\widetilde{\mathrm{Ind}}_{T,W}^V A[d_1-d_2+(n-1)\sum_i \mathrm{Rank}(T_i)\mathrm{Rank}(W_i)], \tag{19}$$

and

$$\mathrm{Res}_{T,W}^V A=\overline{\mathrm{Res}}_{T,W}^V A[d_1-d_2-2\mathrm{dim}G_V/P+(n-1)\sum_i \mathrm{Rank}(T_i)\mathrm{Rank}(W_i)]. \tag{20}$$

Then we have the following corollary.

Corollary 1. $\mathbb{D}(\mathrm{Ind}_{T,W}^V(A))=\mathrm{Ind}_{T,W}^V(\mathbb{D}(A))$.

Proof.

$$\begin{aligned}
\mathbb{D}(\mathrm{Ind}_{T,W}^V(A)) &= \mathbb{D}(\widetilde{\mathrm{Ind}}_{T,W}^V A[d_1-d_2+(n-1)\sum_i \mathrm{Rank}(T_i)\mathrm{Rank}(W_i)])\\
&= \mathbb{D}(\widetilde{\mathrm{Ind}}_{T,W}^V A)[-(d_1-d_2+(n-1)\sum_i \mathrm{Rank}(T_i)\mathrm{Rank}(W_i))]\\
&= \widetilde{\mathrm{Ind}}_{T,W}^V(\mathbb{D}(A))[d_1-d_2+(n-1)\sum_i \mathrm{Rank}(T_i)\mathrm{Rank}(W_i)]\\
&= \mathrm{Ind}_{T,W}^V(\mathbb{D}(A)).
\end{aligned}$$

$\square$

Corollary 2. $\mathrm{Ind}_{T,W}^V(L_{T,\underline{i},\underline{k}}\boxtimes L_{W,\underline{j},\underline{l}})=L_{V,(\underline{i}\ \underline{j}),(\underline{k}\ \underline{l})}$.

Proof. Denote $M:=d_1-d_2+(n-1)\sum_i \mathrm{Rank}(T_i)\mathrm{Rank}(W_i)$, then we have

$$\begin{aligned}
&\mathrm{Ind}_{T,W}^V(L_{T,\underline{i},\underline{k}}\boxtimes L_{W,\underline{j},\underline{l}})\\
&=\widetilde{\mathrm{Ind}}_{T,W}^V(L_{T,\underline{i},\underline{k}}\boxtimes L_{W,\underline{j},\underline{l}})[M]\\
&=\widetilde{\mathrm{Ind}}_{T,W}^V(\widetilde{L}_{T,\underline{i},\underline{k}}\boxtimes \widetilde{L}_{W,\underline{j},\underline{l}})[d(T,\underline{i},\underline{k})+d(W,\underline{j},\underline{l})+(n-1)\sum_i(N_i(\underline{i},\underline{k})+N_i(\underline{j},\underline{l}))+M]\\
&=\widetilde{L}_{V,(\underline{i}\ \underline{j}),(\underline{k}\ \underline{l})}[d(T,\underline{i},\underline{k})+d(W,\underline{j},\underline{l})+M+(n-1)\sum_i(N_i(\underline{i},\underline{k})+N_i(\underline{j},\underline{l}))]\\
&=L_{V,(\underline{i}\ \underline{j}),(\underline{k}\ \underline{l})}[(n-1)\sum_i(N_i(\underline{i},\underline{k})+N_i(\underline{j},\underline{l})-N_i((\underline{i}\ \underline{j}),(\underline{k}\ \underline{l})))+M-d_1+d_2].
\end{aligned}$$

The last equality holds because

$$d(T,\underline{i},\underline{k})+d(W,\underline{j},\underline{l})+d_1-d_2-d(V,(\underline{i}\ \underline{j}),(\underline{k}\ \underline{l}))=0$$

which follows from Remark 1 and Lusztig's argument in 9.2.7 in [23]. Therefore the proposition follows from (18). $\square$

4.4. **Bilinear form.** Let A and B be two G-equivariant semisimple complexes on algebraic variety X. Let's choose an integer m and a smooth irreducible algebraic variety Γ with a free action of G such that $H^i(\Gamma, \overline{\mathbb{Q}}_l) = 0$ for $i = 1, \cdots, m$. G acts diagonally on $\Gamma \times X$. Consider the diagram

$$X \xleftarrow{s} \Gamma \times X \xrightarrow{t} G \setminus (\Gamma \times X)$$

with the obvious projection maps s and t. We have $s^*A = t^*({}_\Gamma A)$ and $s^*B = t^*({}_\Gamma B)$ for well defined semisimple complexes ${}_\Gamma A$ and ${}_\Gamma B$ on ${}_\Gamma X := G \setminus (\Gamma \times X)$.

By the argument in [8, 15], if m is large enough, then

$$\dim H_c^{j+2\dim G-2\dim\Gamma}({}_\Gamma X,\ {}_\Gamma A \otimes {}_\Gamma B) = \dim H_c^j({}_\Gamma X,\ {}_\Gamma A[\dim G \setminus \Gamma] \otimes {}_\Gamma B[\dim G \setminus \Gamma])$$

is independent of m and Γ. Denote this by $d_j(X, G; A, B)$.

Suppose A, A' and B are semisimple G-equivariant complexes on X. Then we have the following properties for $d_j(X, G; A, B)$ (see [8, 14, 15, 23]):

(1) $d_j(X, G; A, B) = d_j(X, G; B, A)$.
(2) $d_j(X, G; A[n], B[m]) = d_{j+n+m}(X, G; A, B)$ for any $m, n \in \mathbb{Z}$.
(3) $d_j(X, G; A \oplus A', B) = d_j(X, G; A, B) + d_j(X, G; A', B)$.
(4) If A and B are perverse sheaves, then so are ${}_\Gamma A[\dim G \setminus \Gamma]$ and ${}_\Gamma B[\dim G \setminus \Gamma]$. Moreover, we have $d_j(X, G; A, B) = 0$ for all $j > 0$. If, in addition, A and B are simple and $B \simeq \mathbb{D}A$, then $d_0(X, G; A, B)$ is 1 and is zero otherwise.
(5) If A' and B' are in $\mathcal{Q}_T^f$ and A'' and B'' are in $\mathcal{Q}_W^f$, then
$d_j(E_T \times E_W, G_T \times G_W; A' \boxtimes A'', B' \boxtimes B'')$
$= \Sigma_{j'+j''=j} d_{j'}(E_T, G_T; A', B') d_{j''}(E_W, G_W; A'', B'')$.
(6) Let $K, K' \in \mathcal{Q}_V^f$. The following two conditions are equivalent:
  (i) $K \simeq K'$;
  (ii) $d_j(E_V, G_V; K, B) = d_j(E_V, G_V; K', B)$ for all simple objects $B \in \mathcal{P}_V^f$ and $j \in \mathbb{Z}$.

Lemma 4 ([8])**.** *Let $A \in \mathcal{Q}_{T,W}^f$ and $B \in \mathcal{Q}_V^f$. Then for any $j \in \mathbb{Z}$,*

$$d_j(E_T \times E_W, G_T \times G_W; A, \overline{\mathrm{Res}}_{T,W}^V B) = d_{j'}(E_V, G_V; \widetilde{\mathrm{Ind}}_{T,W}^V A, B),$$

where $j' = j + 2\dim G_V/P$.

Proposition 8. *Let $A \in \mathcal{Q}_{T,W}^f$ and $B \in \mathcal{Q}_V^f$. Then for any $j \in \mathbb{Z}$,*

$$d_j(E_T \times E_W, G_T \times G_W; A, \mathrm{Res}_{T,W}^V B) = d_j(E_V, G_V; \mathrm{Ind}_{T,W}^V A, B).$$

Proof. This follows directly from definitions (19), (20) and Lemma 4. □

Remark 4. The algebra structure of the composition subalgebra of the Hall algebra associated quivers is independent of the orientation of the given quiver. To give a geometric realization of this subalgebra, one must show that the algebra constructed by using perverse sheaves is also independent of the orientation. The Fourier Deligne transform, later denoted by Φ, is the tool used to prove this. Let $\mathcal{Q}^f = \oplus_V \mathcal{Q}_V^f$, the functors $\mathrm{Ind}_{T,W}^V$ and $\mathrm{Res}_{T,W}^V$ give an algebra and an coalgebra structure of $\mathcal{Q}^f$. With this point of view, one needs to show Φ commutes with $\mathrm{Ind}_{T,W}^V$ and $\mathrm{Res}_{T,W}^V$.

4.5. **Fourier Deligne transform.** In this section, let's consider a new orientation of the given quiver. Denote the source of the arrow h by $s(h) = {}'h$ and its target by $t(h) = {}''h$ for the new orientation. Recall that we denote the source of the arrow h by $s(h) = h'$ and its target by $t(h) = h''$ for the old orientation. Let $H_1 = \{h \in H \mid {}'h = h', {}''h = h''\}$ and $H_2 = \{h \in H \mid {}'h = h'', {}''h = h'\}$. For a given I-graded free R-module V, denote

$$E_V = \oplus_{h\in H_1} \mathrm{Hom}_R(V_{h'}, V_{h''}) \oplus \oplus_{h\in H_2} \mathrm{Hom}_R(V_{h'}, V_{h''}),$$

$$'E_V = \oplus_{h\in H_1} \mathrm{Hom}_R(V_{h'}, V_{h''}) \oplus \oplus_{h\in H_2} \mathrm{Hom}_R(V_{h''}, V_{h'}),$$

and

$$\dot{E}_V = \oplus_{h\in H_1} \mathrm{Hom}_R(V_{h'}, V_{h''}) \oplus \oplus_{h\in H_2} \mathrm{Hom}_R(V_{h'}, V_{h''}) \oplus \oplus_{h\in H_2} \mathrm{Hom}_R(V_{h''}, V_{h'}).$$

Then we have the natural projection maps

$$E_V \xleftarrow{s} \dot{E}_V \xrightarrow{t} {}'E_V.$$

Consider E_V^R as a subset of E_V^k, then we can define a map $\mathcal{T}_V : \dot{E}_V \to k$ by

$$\mathcal{T}_V(a,b,c) = \sum_{h\in H_2} tr(V_{h'} \xrightarrow{b} V_{h''} \xrightarrow{c} V_{h'}), \tag{21}$$

where tr is the trace function of the endomorphism of k-vector space. Clearly, $\mathcal{T}_V$ is a bilinear map.

Define

$$\begin{aligned}\Phi : \mathcal{D}(E_V) &\to \mathcal{D}('E_V)\\ A &\mapsto t_!(s^*(A) \otimes L_{\mathcal{T}_V})[d_V],\end{aligned}$$

where $d_V = \dim(\oplus_{h\in H_2} \mathrm{Hom}_R(V_{h'}, V_{h''})) = n \sum_{h\in H_2} \mathrm{Rank}(V_{h'})\mathrm{Rank}(V_{h''})$.

Similarly, we have the projection maps

$$E_T \times E_W \xleftarrow{\overline{s}} \dot{E}_T \times \dot{E}_W \xrightarrow{\overline{t}} {}'E_T \times {}'E_W.$$

Define $\overline{\mathcal{T}} : \dot{E}_T \times \dot{E}_W \to k$ by $\overline{\mathcal{T}} := \mathcal{T}_T + \mathcal{T}_W$, where $\mathcal{T}_T : \dot{E}_T \to k$ (resp. $\mathcal{T}_W : \dot{E}_W \to k$) is defined in (21) replacing V by T (resp. W). In a similar fashion, one can define

$$\begin{aligned}\Phi : \mathcal{D}(E_T \times E_W) &\to \mathcal{D}('E_T \times {}'E_W)\\ A &\mapsto \overline{t}_!(\overline{s}^*(A) \otimes L_{\overline{\mathcal{T}}})[d_T + d_W].\end{aligned}$$

Proposition 9. *For any $B \in \mathcal{Q}_V^f$, we have*

$$\Phi\overline{\mathrm{Res}}^V_{T,W}(B) = \overline{\mathrm{Res}}^V_{T,W}\Phi(B)[\pi],$$

where $\pi = n\Sigma_{h\in H_2}(\mathrm{Rank}(T_{h''})\mathrm{Rank}(W_{h'}) - \mathrm{Rank}(T_{h'})\mathrm{Rank}(W_{h''}))$.

Proof. The following proof is based on Lusztig's proof for Proposition 10.1.2 in [23]. Consider the following diagram,

$$\begin{tikzcd}
E_T \times E_W & F \arrow[l, "p"'] \arrow[rrr, "\iota"] & & & E_V \\
\dot{E}_T \times \dot{E}_W \arrow[u, "\overline{s}"] \arrow[d, "\overline{t}"'] & \psi \arrow[l, "\dot{p}"'] \arrow[u, "\dot{s}"'] & \dot{F} \arrow[l, "\dot{q}"'] \arrow[r, "\dot{\xi}"] & \Xi \arrow[r, "i"] \arrow[d, "\dot{t}"'] & \dot{E}_V \arrow[u, "s"'] \arrow[d, "t"] \\
{}'E_T \times {}'E_W & & & {}'F \arrow[lll, "{}'p"'] \arrow[r, "{}'\iota"] & {}'E_V.
\end{tikzcd}$$

Here $F = \{x \in E_V \mid x_h(W_{h'}) \subset W_{h''},\ \forall h \in H\}$;

${}'F = \{x \in {}'E_V \mid x_h(W_{'h}) \subset W_{''h},\ \forall h \in H\}$;

$\dot{F} = \Big\{(x_1, x_2, x_3) \in \dot{E}_V \mid (x_1, x_2) \in F,\ (x_1, x_3) \in\ {}'F\Big\}$, where $x_1 \in \oplus_{h\in H_1} \mathrm{Hom}_R(V_{h'}, V_{h''})$, $x_2 \in \oplus_{h\in H_2} \mathrm{Hom}_R(V_{h'}, V_{h''})$, $x_3 \in \oplus_{h\in H_2} \mathrm{Hom}_R(V_{h''}, V_{h'})$. Similarly, in the rest of proof, the subscript $1, 2, 3$ always mean the first, second and third component in $\dot{E}_V$ or its subset respectively and super-script T (resp. W) means the object is in $\dot{E}_T$ (resp. $\dot{E}_W$).

$\Xi = \Big\{(y_1, y_2, y_3) \in \dot{E}_V \mid (y_1, y_3) \in {}'F\Big\}$;

$\psi = \Big\{(x, y^T, y^W) \in F \times \dot{E}_T \times \dot{E}_W \mid x'' = (y_1^T, y_2^T),\ x' = (y_1^W, y_2^W)\Big\}$, where $x' = x_W, x'' = x_T$ and in the rest of proof, single prime always means restriction to W and double prime means the induce map $V/W \to V/W$. Maps are defined as follows.

$\dot{s} : \psi \to F,\ (x, y^T, y^W) \mapsto x$;

$\dot{p} : \psi \to \dot{E}_T \times \dot{E}_W,\ (x, y^T, y^W) \mapsto (x', y_3^T, x'', y_3^W)$;

$\dot{q} : \dot{F} \to \psi,\ (x_1, x_2, x_3) \mapsto ((x_1, x_2), (x_1', x_2', x_3'), (x_1'', x_2'', x_3''))$;

$i : \Xi \to \dot{E}_V$ is an embedding;

$\dot{t} : \Xi \to {}'F, (y_1, y_2, y_3) \mapsto (y_1, y_3)$;

$\dot{\xi} : \dot{F} \to \Xi,\ (y_1, y_2, y_3) \mapsto (y_1, y_2, y_3)$.

Let $Z = \{(y_1, y_2, y_3) \in \Xi \mid y_3(W_{h''}) = 0, y_3(T_{h''}) = 0, y_1 = y_2 = 0\}$ and let $c : \Xi \to \Xi/Z$ be the canonical projection map. Define $\widetilde{\mathcal{T}} : \Xi \to k$ sending x to $\mathcal{T}_V i(x)$. It is clear that $\widetilde{\mathcal{T}}|_Z = 0$. Let $\widetilde{\mathcal{T}}_1 : \Xi/Z \to k$ be the induce map of $\widetilde{\mathcal{T}}$.

We are going to show $\widetilde{\mathcal{T}}_1$ is constant if and only if $tr(y_3 x_2) = 0$ for all y_3.

In fact, let $\overline{x} = x + y, x \in \Xi, y \in Z$, then

$$\begin{aligned}
\widetilde{\mathcal{T}}(\overline{x}) &= \mathcal{T}_V(i(\overline{x})) \\
&= \mathcal{T}_V((x + y)_1, (x + y)_2, (x + y)_3) \\
&= \mathcal{T}_V(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\
&= \Sigma_{h\in H_2} tr((x_3 + y_3)(x_2 + y_2)) \\
&= \Sigma_{h\in H_2} tr(x_3 x_2 + x_3 y_2 + y_3 x_2 + y_3 y_2) \\
&= \Sigma_{h\in H_2} tr(x_3 x_2) + tr(y_3 x_2).
\end{aligned}$$

Since $\widetilde{\mathcal{T}}_1$ is affine linear function, $\widetilde{\mathcal{T}}_1$ is constant if and only if $tr(y_3 x_2) = 0$ for all y_3.

Next we want to show that $tr(y_3 x_2) = 0$ for all y_3 satisfying $y_3(W_{h''}) = 0, y_3(T_{h''}) = 0$ if and only if $\{x + y \mid \forall y \in Z\} \subseteq \dot{\xi}(\dot{F})$, i.e. $(x_1, x_2) \in F$.

Since $V_{h''}, W_{h''}$ and $T_{h''}$ are all free modules, we can fix a basis of $W_{h''}$ and extend it to a basis of $V_{h''}$. Under this basis, we have a block matrix decomposition $y_3 = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}$. Here $*$ is any block. For any $x_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\dot{t}ry_3x_2 = 0$ if and only if $tr(*c) = 0$. Since $*$ is an arbitrary matrix, this is true if and only if $c = 0$. Therefore $(x_1, x_2) \in F$.

Now let $\Xi' = \Xi - \dot{\xi}(\dot{F})$. Denote $c' = c|_{\Xi'}$ and $\mathcal{T}' = \widetilde{\mathcal{T}}|_{\Xi'}$. Then the restriction of $\mathcal{T}'$ to any fibre of c' is a non-constant affine linear function. Hence by Section 2.4(2), the local system $\mathcal{L}_{\mathcal{T}'}$ on Ξ' satisfies $c'_!(\mathcal{L}_{\mathcal{T}'}) = 0$.

Since $\dot{\xi} : \dot{F} \to \Xi$ is a closed embedding, applying 2.2.3(4) to the partition $\Xi = \Xi' \cup \dot{\xi}(\dot{F})$, we have a distinguished triangle

$$c_! j_! j^* \mathcal{L}_{\widetilde{\mathcal{T}}} \longrightarrow c_! \mathcal{L}_{\widetilde{\mathcal{T}}} \longrightarrow c_! \dot{\xi}_! (\dot{\xi}^* \mathcal{L}_{\widetilde{\mathcal{T}}}) \xrightarrow{[1]},$$

where $j : \Xi' \to \Xi$ is the open embedding. By the above argument, $c_! j_! j^* \mathcal{L}_{\widetilde{\mathcal{T}}} = 0$. Therefore $c_! \dot{\xi}_! \dot{\xi}^* \mathcal{L}_{\widetilde{\mathcal{T}}} = c_! \mathcal{L}_{\widetilde{\mathcal{T}}}$

Clearly the composition $si : \Xi \to E_V, (x_1, x_2, x_3) \mapsto (x_1, x_2)$ factors through Ξ/Z since $c : \Xi \to \Xi/Z$ sends (x_1, x_2, x_3) to $(x_1, x_2, \overline{x}_3)$. Let $si = gc$, where $g : \Xi/Z \to E_V$. By projection formula, we have

$$c_!(\dot{\xi}_! \dot{\xi}^* \mathcal{L}_{\widetilde{\mathcal{T}}} \otimes c^* g^* B) = c_! \dot{\xi}_! \dot{\xi}^* \mathcal{L}_{\widetilde{\mathcal{T}}} \otimes g^* B = c_! \mathcal{L}_{\widetilde{\mathcal{T}}} \otimes g^* B = c_!(\mathcal{L}_{\widetilde{\mathcal{T}}} \otimes c^* g^* B).$$

Therefore,

$$c_!(\dot{\xi}_! \dot{\xi}^* \mathcal{L}_{\widetilde{\mathcal{T}}} \otimes i^* s^* B) = c_!(\mathcal{L}_{\widetilde{\mathcal{T}}} \otimes i^* s^* B).$$

The composition $'p\dot{t} : \Xi \to {'E_T} \times {'E_W}, (x_1, x_2, x_3) \mapsto (x'_1, x'_3, x''_1, x''_3)$ also factor through Ξ/Z. Because if $\overline{z}_3 = \overline{x}_3$, then there exists $y_3 \in Z$ such that $x_3 - z_3 = y_3$. Hence $(x_3 - z_3)(W_{h''}) = 0$ and $(x_3 - z_3)(T_{h''}) = 0$, i.e. $x_3(a) = z_3(a), \forall a \in W_{h''}$ and $a \in T_{h''}$. Therefore $x''_3 = z''_3, x'_3 = z'_3$. So

$$'p_! \dot{t}_! (\dot{\xi}_! (\dot{\xi}^* \mathcal{L}_{\widetilde{\mathcal{T}}}) \otimes i^* s^* B) = {'p_!} \dot{t}_! (\mathcal{L}_{\widetilde{\mathcal{T}}} \otimes i^* s^* B).$$

Since $\mathcal{T}_V i \dot{\xi} = \overline{\mathcal{T}} \dot{p} \dot{q}$, we have $\dot{q}^* \dot{p}^* \mathcal{L}_{\overline{\mathcal{T}}} = \dot{\xi}^* i^* \mathcal{L}_{\mathcal{T}_V} = \dot{\xi}^* \mathcal{L}_{\widetilde{\mathcal{T}}}$.

Since $\dot{q}$ is a vector bundle with rank $m = n \sum_{h \in H_2} \mathrm{Rank}(T_{h'}) \mathrm{Rank}(W_{h''})$, we have $\dot{q}_! \dot{q}^* L = L[-2m]$ for all $L \in \mathcal{D}(\psi)$. Therefore,

$$\begin{aligned}
\Phi(\overline{\mathrm{Res}}^V_{T,W} B) &= \bar{t}_! (\mathcal{L}_{\overline{\mathcal{T}}} \otimes \overline{s}^* p_! \iota^* B)[d_T + d_W] \\
&= \bar{t}_! (\mathcal{L}_{\overline{\mathcal{T}}} \otimes \dot{p}_! \dot{s}^* \iota^* B)[d_T + d_W] \\
&= \bar{t}_! (\mathcal{L}_{\overline{\mathcal{T}}} \otimes \dot{p}_! \dot{q}_! \dot{q}^* \dot{s}^* \iota^* B[2m])[d_T + d_W] \\
&= \bar{t}_! \dot{p}_! \dot{q}_! (\dot{q}^* \dot{p}^* (\mathcal{L}_{\overline{\mathcal{T}}}) \otimes \dot{q}^* \dot{s}^* \iota^* B)[2m + d_T + d_W] \\
&= {'p_!} \dot{t}_! \dot{\xi}_! (\dot{q}^* \dot{p}^* (\mathcal{L}_{\overline{\mathcal{T}}}) \otimes \dot{\xi}^* i^* s^* B)[2m + d_T + d_W] \\
&= {'p_!} \dot{t}_! (\dot{\xi}_! (\dot{q}^* \dot{p}^* \mathcal{L}_{\overline{\mathcal{T}}}) \otimes i^* s^* B)[2m + d_T + d_W] \\
&= {'p_!} \dot{t}_! (\dot{\xi}_! \dot{\xi}^* \mathcal{L}_{\widetilde{\mathcal{T}}} \otimes i^* s^* B)[2m + d_T + d_W] \\
&= {'p_!} \dot{t}_! (\mathcal{L}_{\widetilde{\mathcal{T}}} \otimes i^* s^* B)[2m + d_T + d_W]
\end{aligned}$$

and

$$\begin{aligned}\overline{\mathrm{Res}}^{V}_{T,W}(\Phi(B))[\pi] &= {}'p_!\, {}'\iota^* t_!(\mathcal{L}_{\mathcal{T}} \otimes s^*B)[\pi + d_V] \\ &= {}'p_! \dot{t}_! i^*(\mathcal{L}_{\mathcal{T}} \otimes s^*B)[\pi + d_V] \\ &= {}'p_! \dot{t}_!(\mathcal{L}_{\widetilde{\mathcal{T}}} \otimes i^*s^*B)[\pi + d_V].\end{aligned}$$

Using $\mathrm{Rank}(V_i) = \mathrm{Rank}(T_i) + \mathrm{Rank}(W_i)$, we have $\pi + d_V = 2m + d_T + d_W$. This finishes the proof. $\square$

Lemma 5. *$\Phi(\widetilde{L}^f_{V,\underline{i},\underline{k}}) = {}'\widetilde{L}^f_{V,\underline{i},\underline{k}}[M]$ for some M.*

Proof. This proof is based on Lusztig's proof for 10.2.2 in [23]. Consider the following diagram,

$$\begin{tikzcd} \widetilde{\mathcal{F}}^f_{V,\underline{i},\underline{k}} \arrow[d, "\pi"'] & \Xi \arrow[l, "b"'] \arrow[r, "c"] \arrow[d, "\rho"'] & \Xi' \arrow[d, "p"] \\ E_V & \dot{E}_V \arrow[l, "s"'] \arrow[r, "t"] & {}'E_V. \end{tikzcd}$$

Here

$$\Xi = \left\{(x, y, \mathfrak{f}) \in E_V \times {}'E_V \times \mathcal{F}^f_{V,\underline{i},\underline{k}} \mid \mathfrak{f} \text{ is an } x-\text{stable and } y_h = x_h, \forall h \in H_2\right\},$$

and

$$\Xi' = \left\{(y, \mathfrak{f}) \in {}'E_V \times \mathcal{F}^f_{V,\underline{i},\underline{k}} \mid y_h(V^l_{h'}) \subset V^l_{h''}, \forall l \text{ and } h \in H_1\right\}.$$

Maps are defined as follows. $b(x, y, \mathfrak{f}) = (z, \mathfrak{f}), c(x, y, \mathfrak{f}) = (y, \mathfrak{f})$ and π, ρ, p are obvious projection maps. Then the left square is a cartesian square and the right square is commutative. By the definition of $\widetilde{L}^f_{V,\underline{i},\underline{k}}$, we have

$$\Phi(\widetilde{L}^f_{V,\underline{i},\underline{k}}) = t_!(\mathcal{L}_T \otimes s^*\pi_!\mathbf{1})[d_V] = t_!(\mathcal{L}_T \otimes \rho_!\mathbf{1})[d_V].$$

By projection formula in Section 2.2.5,

$$t_!(\mathcal{L}_T \otimes \rho_!\mathbf{1})[d_V] = t_!\rho_!(\rho^*\mathcal{L}_T \otimes \mathbf{1})[d_V] = p_!c_!(\mathcal{L}_{T'})[d_V].$$

The last equality follows from $T' = T\rho$.

Let $\Xi_0 = \{(x, y, \mathfrak{f}) \in \Xi \mid \mathfrak{f} \text{ is } y-\text{stable}\}$ and $\Xi_1 = \Xi - \Xi_0$. Clearly, $T'|_{\Xi_1}$ is not a constant function. By Property 2.4(2), $c_!(\mathcal{L}_{T'}|_{\Xi_1}) = 0$. Since $j : \Xi_1 \to \Xi$ is an open embedding, $c_!j_!j^*\mathcal{L}_{T'} = 0$. Applying 2.2.3(4) to the partition $\Xi = \Xi_1 \coprod \Xi_0$, we have a distinguish triangle,

$$c_!j_!j^*\mathcal{L}_{T'} \longrightarrow c_!\mathcal{L}_{T'} \longrightarrow c_!i_!i^*\mathcal{L}_{T'} \xrightarrow{[1]},$$

where $i : \Xi_0 \to \Xi$ is the closed embedding. Then $c_!\mathcal{L}_{T'} = c_!i_!i^*\mathcal{L}_{T'}$.

For any $(x, y, \mathfrak{f}) \in \Xi_0$,

$$T'(x, y, \mathfrak{f}) = T(x, y) = \sum_{h \in H_2} tr(y_h x_h : V_{h'} \to V_{h'}).$$

Let $\mathfrak{f} = g_{\mathfrak{f}}\mathfrak{f}_0$ for some $g_{\mathfrak{f}} \in H/(H \bigcap P^R)$ (see Remark 2). Since $\mathfrak{f}_0$ is stable under both x and y, $\mathfrak{f}$ is stable under both $g_{\mathfrak{f}}^{-1} \cdot x$ and $g_{\mathfrak{f}}^{-1} \cdot y$. Since $(g_{\mathfrak{f}}^{-1} \cdot y)_h (g_{\mathfrak{f}}^{-1} \cdot x)_h = (g_{\mathfrak{f}}^{-1})_{h'} y_h x_h (g_{\mathfrak{f}})_{h'}$, $tr(y_h x_h) = tr((g_{\mathfrak{f}}^{-1})_{h'} y_h x_h (g_{\mathfrak{f}})_{h'})$. Moreover, we have

$$tr((g_{\mathfrak{f}}^{-1})_{h'} y_h x_h (g_{\mathfrak{f}})_{h'} : V_{h'} \to V_{h'}) = \sum_l tr((g_{\mathfrak{f}}^{-1})_{h'} y_h x_h (g_{\mathfrak{f}})_{h'} : V_{h'}^{l-1}/V_{h'}^l \to V_{h'}^{l-1}/V_{h'}^l).$$

Since V^{l-1}/V^l concentrate on one vertex, for any l, at least one of $V_{h'}^{l-1}/V_{h'}^l$ and $V_{h''}^{l-1}/V_{h''}^l$ is zero. Therefore, $tr(y_h x_h : V_{h'} \to V_{h'}) = 0$ for each $h \in H_2$. i.e. $T'(x, y, \mathfrak{f}) = 0$. Hence $\mathcal{L}_{T'}|_{\Xi_0} = \mathbf{1}$. i.e. $i_! i^* \mathcal{L}_{T'} = \mathbf{1}$. Therefore,

$$p_! c_! (\mathcal{L}_{T'})[d_V] = p_! c_! i_! i^* \mathcal{L}_{T'}[d_V] = p_! (c|_{\Xi_0})_! \mathbf{1}[d_V].$$

Since $c(\Xi_0) = \widetilde{\mathcal{F}}^f_{V,\underline{i},\underline{k}}$ and $c|_{\Xi_0}$ is a vector bundle. Denote the rank of $c|_{\Xi_0}$ by M'. Then

$$p_! (c|_{\Xi_0})_! \mathbf{1}[d_V] = p_! \mathbf{1}[d_V - 2M'].$$

Since $p|_{'\widetilde{\mathcal{F}}^f_{V,\underline{i},\underline{k}}} = {}'\pi$, where ${}'\pi : {}'\widetilde{\mathcal{F}}^f_{V,\underline{i},\underline{k}} \to {}'E_V$ is the first projection map,

$$p_! \mathbf{1}[d_V - 2M'] = {}'\widetilde{L}^f_{V,\underline{i},\underline{k}}[d_V - 2M'].$$

Let $M = d_V - 2M'$. The proposition follows. $\square$

Corollary 3. $\Phi(\widetilde{\mathrm{Res}}^V_{T,W}(B)) = \widetilde{\mathrm{Res}}^V_{T,W}(\Phi(B))[\pi]$.

Proof. From Lemma 5, $\Phi(\mathcal{Q}_V^f) \subset {}'\mathcal{Q}_V^f$, where ${}'\mathcal{Q}_V^f$ is defined similarly as $\mathcal{Q}_V^f$ for ${}'E_V$. By the same argument $\Phi({}'\mathcal{Q}_V^f) \subset \mathcal{Q}_V^f$. Since $\Phi(\Phi(K)) = K$ (see 10.2.3 in [23]), for any $K \in \mathcal{Q}_V^{Nf} \setminus \mathcal{Q}_V^f$, if $\Phi(K) \in {}'\mathcal{Q}_V^f$, then $K = \Phi(\Phi(K)) \in \mathcal{Q}_V^f$. This is a contradiction. Therefore, $\Phi(K) \notin {}'\mathcal{Q}_V^f$ for any $K \in \mathcal{Q}_V^{Nf} \setminus \mathcal{Q}_V^f$.

By definition of $\widetilde{\mathrm{Res}}^V_{T,W}$, the corollary follows from Proposition 9. $\square$

Corollary 4. $\Phi(\mathrm{Res}^V_{T,W}(B)) = \mathrm{Res}^V_{T,W}(\Phi(B))$.

Proof. From (20) and Corollary 3, it is enough to show $\pi + d_1 - {}'d_1 = 0$, where ${}'d_1$ is defined similarly as d_1 for the new orientation. Recall $d_1 = \dim G_V/U + n \sum_{h \in H} \mathrm{Rank}(T_{h'})\mathrm{Rank}(W_{h''})$. Since $\dim G_V/U$ has nothing to do with orientations and $H = H_1 \bigcup H_2$, it is enough to show

$$\pi = \sum_{h \in H_2} \mathrm{Rank}(T_{h'})\mathrm{Rank}(W_{h''}) - \sum_{h \in {}'H_2} \mathrm{Rank}(T_{h'})\mathrm{Rank}(W_{h''}).$$

Here ${}'H_2$ is the set of all arrows with opposite orientation of arrows in H_2. The corollary follows that $\sum_{h \in {}'H_2} \mathrm{Rank}(T_{h'})\mathrm{Rank}(W_{h''}) = \sum_{h \in H_2} \mathrm{Rank}(T_{h''})\mathrm{Rank}(W_{h'})$. $\square$

Lemma 6. *Let $A \in \mathcal{Q}_V$, $A' \in {}'\mathcal{Q}_V$, then for any $j \in \mathbb{Z}$, we have*

$$d_j(E_V, G_V; A, \Phi(A')) = d_j({}'E_V, G_V; \Phi(A), A').$$

Proof. Let u (resp. $u', \dot{u}$) be the map of ${}_\Gamma E_V$ (resp. ${}_\Gamma {}'E_V$, ${}_\Gamma(E_V \times {}'E_V)$) to the point (see Section 4.4 for notations). By definition of $d_j(E_V, G_V; A, A')$, we have

$$d_j(E_V, G_V; A, \Phi(A')) = \dim H^{j + 2\dim G - 2\dim \Gamma}(\mathrm{pt}, u_!({}_\Gamma A \otimes {}_\Gamma \Phi(A'))).$$

The lemma follows from Section 2.4(1). $\square$

Corollary 5. $\Phi(\mathrm{Ind}^V_{T,W}(B)) = \mathrm{Ind}^V_{T,W}(\Phi(B))$

Proof. By Proposition 4.4(6), it is enough to prove

$$d_j('E_V, G_V; \Phi(\mathrm{Ind}^V_{T,W}(B)), \Phi K) = d_j('E_V, G_V; \mathrm{Ind}^V_{T,W}(\Phi(B)), \Phi K)$$

for all simple objects $K \in \mathcal{P}^f_V$ and $j \in \mathbb{Z}$.

By Lemma 6 and Proposition 8,

$$\begin{aligned} &d_j('E_V, G_V; \Phi(\mathrm{Ind}^V_{T,W}(B)), \Phi(K)) \\ &\quad = d_j(E_V, G_V; (\mathrm{Ind}^V_{T,W}(B)), K) \\ &\quad = d_j(E_T \times E_W, G_T \times G_W; B, \mathrm{Res}^V_{T,W}(K)). \end{aligned}$$

By Proposition 8,

$$\begin{aligned} &d_j('E_V, G_V; \mathrm{Ind}^V_{T,W}(\Phi(B)), \Phi(K)) \\ &\quad = d_j('E_T \times 'E_W, G_T \times G_W; \Phi(B), \mathrm{Res}^V_{T,W}(\Phi(K))) \\ &\quad = d_j(E_T \times E_W, G_T \times G_W; B, \Phi(\mathrm{Res}^V_{T,W}(\Phi(K)))) \text{ (by Lemma 6)} \\ &\quad = d_j(E_T \times E_W, G_T \times G_W; B, \mathrm{Res}^V_{T,W}(K)) \text{ (by Corollary 4).} \end{aligned}$$

□

4.6. **Additive generators.** In this section, we fix a vertex $i \in I$ and assume W is an I-grade free R-submodule of V such that $T = V/W$ is also a free R-module.

Remark 5. By the Fourier-Deligne transform, we can assume W satisfy that $W_{h'} = V_{h'}$, $\forall h \in H$ and by induction we can further assume $\mathrm{Supp}(T) = \{i\}$. Hence, $E_T = 0$, and $E_W \simeq F$.

Given any matrix X with entries in R, any k-th minor D_k of X can be written into $D_k(X) = f_{i0}(X) + f_{i1}(X)t + \cdots + f_{ir}(X)t^r$. We will use super-script to distingue the different k-th minors and their coefficients. For example, $D^r_k(X)$ and $f^r_{kl}(X)$. Note that we take all minors with value in $k[t]$ but not those in R since we are studying the coordinate ring of the k-variety E_V.

If we fix an R-basis for each V_i, then all x_h can be written as a matrix, denoted by X_h, with entries in R. Moreover, $\sum_{h\in H, h''=i} x_h$ corresponds to the matrix $X_i := (X_{h_1}, X_{h_2}, \cdots, X_{h_s})$, where each subscript h_j is an arrow with target vertex i. Given $i \in I$, let

$$B_{V,i,k} = \{x \in E_V \mid D^r_k(X_i) = 0 \text{ for all } r\}.$$

Notice that X_i depends on the choice of basis of V_i, but $B_{V,i,k}$ doesn't depend on the choice of basis of V_i. Because equivalent transformations of matrixes change a k-th minor into another k-th minor which is obtained by multiplying by an invertible element in R. Moreover, $B_{V,i,k}$ is a closed subset of E_V. Given $(k,l) \in \mathbb{N} \times \mathbb{N}$, let

$$C_{V,i,(k,l)} = \{x \in E_V \mid f^r_{ks}(X_i) = 0 \text{ for all } r, \text{ and all } s \le l\}.$$

By the same reason, this set doesn't depend on the choice of basis of V_i and it is a closed subset of E_V.

Now define a total order on $\mathbb{N} \times \mathbb{N}$ by

$$(k,l) < (r,s) \text{ if and only if } k < r \text{ or } k = r, s < l.$$

Let $E_{V,i,\leq(k,l)} = C_{V,i,(k,l)} \bigcap B_{V,i,k+1}$. This is a closed subset. It is clear that

$$B_{V,i,k} \subset \cdots \subset E_{V,i,\leq(k,l)} \subset E_{V,i,\leq(k,l-1)} \subset \cdots \subset B_{V,i,k+1} \subset \cdots \subset E_V. \tag{22}$$

Furthermore, for $x \in E_V$, there exists (k,l) such that $x \in E_{V,i,\leq(k,l)}$.

Let $E_{V,i,(k,l)} = E_{V,i,\leq(k,l)} \backslash E_{V,i,\leq(k,l+1)}$. This is a locally closed subset of E_V and its closure $\overline{E_{V,i,(k,l)}} = E_{V,i,\leq(k,l)}$. From linear algebra, $E_{V,i,(k,l)}$ is stable under G_V-action.

Recall $p : G_V \times^P E_W \to E_V$ is a G_V-equivariant map sending (g,x) to $g\iota(x)$, where $\iota : E_W \to E_V$ is an embedding. Let $p_0 := p|_{G_V \times^P E_{W,i,(k,l)}} : G_V \times^P E_{W,i,(k,l)} \to E_{V,i,(k,l)}$.

Lemma 7. *p_0 is a vector bundle with rank $d_0 = (v_i - w_i)(l + n(w_i - k))$, where $v_i = \mathrm{Rank(V_i)}$ and $w_i = \mathrm{Rank(W_i)}$.*

Proof. For any $y \in E_{V,i,(k,l)}$, $p_0^{-1}(y) = \{(g,x) \mid g\iota(x) = y\}$.

If $g_1\iota(x_1) = g_2\iota(x_2) = y$, then $g_1^{-1}g_2\iota(x_2) = \iota(x_1)$. This implies that x_1 is equivalent to x_2. i.e. there exists $g \in G_W$ such that $gx_1 = x_2$. Hence there exists $h \in P$ such that $h\iota(x_1) = \iota(x_2)$. Then $g_1\iota(x_1) = g_2h\iota(x_1)$. This means that $s := g_1^{-1}g_2h$ is in the stabilizer, $\mathrm{Stab}_{G_V}(\iota(x_1))$, in G_V of $\iota(x_1)$. i.e. $g_2 = g_1sh^{-1}$ for some $s \in \mathrm{Stab}_{G_V}(\iota(x_1))$ and $h \in P$. Hence $(g_1, x_1) \in p_0^{-1}(y)$ if and only if $(g_1sh^{-1}, hx_1) \in p_0^{-1}(y)$ for some $s \in \mathrm{Stab}_{G_V}(\iota(x_1))$. Therefore, $\dim(p_0^{-1}(y)) = \dim(\mathrm{Stab}_{G_V}(\iota(x_1))) - \dim(P \bigcap \mathrm{Stab}_{G_V}(\iota(x_1)))$.

If we fix a basis of W and extend it to a basis of V, then

$$\dim(\mathrm{Stab}_{G_V}(\iota(x_1)) = \dim\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}\right\}),$$

and

$$\dim(P \bigcap \mathrm{Stab}_{G_V}(\iota(x_1)) = \dim\left\{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}\right\}).$$

Therefore,

$$\dim(p_0^{-1}(y)) = \dim(\{c \in \mathrm{Hom}_R(W_i, T_i) \mid cx_1 = 0\}) \tag{23}$$

Given $\underline{j} = (j_0, j_1, \cdots, j_{n-1})$, let

$$M_{i,\underline{j}} = \left\{x \in E_V \mid X_i \simeq \mathrm{Diag}(1^{\oplus j_0}, t^{\oplus j_1}, \cdots, (t^{n-1})^{\oplus j_{n-1}})\right\}.$$

If $y \in M_{i,\underline{j}}$ for some $\underline{j}$ with $|\underline{j}| := \sum_r j_r = k$. Then $y \in E_{V,i,(k,\lambda(\underline{j}))}$, where $\lambda(\underline{j}) = \sum_r rj_r$. From the above argument,

$$\dim(p_0^{-1}(y)) = (v_i - w_i)(\lambda(\underline{j}) + n(w_i - |\underline{j}|))$$

which only depends on $\lambda(\underline{j})$ and $|\underline{j}|$.

On the other hand, from linear algebra, if $y \in E_{V,i,(k,l)}$, then $y \in M_{i,\underline{j}}$ such that $|\underline{j}| = k$ and $\lambda(\underline{j}) = l$. i.e. the dimension of fibers of any element in $E_{V,i,(k,l)}$ only depends on k and l. By (23), p_0 has a vector bundle structure. This finishes proof. ☐

Let $\iota_1 : E_{V,i,\leq(k,l)} \to E_V$ be the closed embedding. Applying base change formula to the following cartesian square,

(24)
$$
\begin{tikzcd}
\widetilde{F}' \arrow[r] \arrow[d, "\pi'"] & \widetilde{\mathcal{F}}_{V,\underline{i},\underline{k}} \arrow[d, "\pi"] \\
E_{V,i,\leq(k,l)} \arrow[r, "\iota_1"] & E_V,
\end{tikzcd}
$$

we have

$$\iota_1^* \widetilde{L}_{V,\underline{i},\underline{k}} = \iota_1^* \pi_! \mathbf{1} = \pi'_! \mathbf{1}.$$

Remark 6. For any simple perverse sheaf $A \in \mathcal{P}_V^f$ (see Section 4.1), i.e. A is a direct summand of $\widetilde{L}_{V,\underline{i},\underline{k}}$ for some $(\underline{i}, \underline{k})$ up to shift, $\iota_1^* A$ is a direct summand of $\iota_1^* \widetilde{L}_{V,\underline{i},\underline{k}} = \pi'_! \mathbf{1}$ up to shift. By the same argument as we show Proposition 3, $\iota_1^* A[d_A]$ is a perverse sheaf on $E_{V,i,\leq(k,l)}$ for some d_A.

Let $\mathcal{P}^f_{V,i,(k,l)}$ be the full subcategory of $\mathcal{M}(E_{V,i,\leq(k,l)})$ consisting of direct sums of perverse sheaves $\iota_1^* A$ up to shifts for some $A \in \mathcal{P}_V^f$. Let

$$\mathcal{P}^{f0}_{V,i,(k,l)} = \{B \in \mathcal{P}^f_{V,i,(k,l)} \mid \mathrm{Supp}(B) \bigcap E_{V,i,(k,l)} \neq \emptyset\}$$

and

$$\mathcal{P}^{f1}_{V,i,(k,l)} = \{B \in \mathcal{P}^f_{V,i,(k,l)} \mid \mathrm{Supp}(B) \bigcap E_{V,i,(k,l)} = \emptyset\}.$$

Then any object $A \in \mathcal{P}^f_{V,i,(k,l)}$ can be decomposed into $A = A^0 \oplus A^1$, where $A^0 \in \mathcal{P}^{f0}_{V,i,(k,l)}$ and $A^1 \in \mathcal{P}^{f1}_{V,i,(k,l)}$. Furthermore, if we require A^1 is the maximal subobject of A in $\mathcal{P}^{f1}_{V,i,(k,l)}$, then such decomposition is unique since A is a semisimple perverse sheaf.

One can similarly define $\mathcal{P}^f_{W,i,(k,l)}$ (resp. $\mathcal{P}^{f0}_{W,i,(k,l)}$ and $\mathcal{P}^{f1}_{W,i,(k,l)}$).

Now consider the following diagrams,

(25)
$$E_{W,i,\leq(k,l)} \xleftarrow{p'_1} G_V \times^U E_{W,I,\leq(k,l)} \xrightarrow{p'_2} G_V \times^P E_{W,i,\leq(k,l)} \xrightarrow{p'_3} E_{V,i,\leq(k,l)}$$

(26)
$$E_{W,i,\leq(k,l)} \xrightarrow{\iota'} E_{V,i,\leq(k,l)}.$$

Here G_V, U, P are the same as in Section 4.3 and the maps are defined similarly as in the Section 4.3 and 4.2. Define the functors $\widetilde{\mathrm{Ind}}^V_{T,W,i,(k,l)}(A) = p'_{3!} p'_{2\flat} p'^*_1(A)$ and $\widetilde{\mathrm{Res}}^V_{T,W.i,(k,l)}(B) = \iota'^*(B)$. This is the induction (resp. restriction) functor defined on $E_{W,i,\leq(k,l)}$ (resp. $E_{V,i,\leq(k,l)}$) instead of E_W (resp. E_V).

Let

(27)
$$\mathrm{Ind}^V_{T,W,i,(k,l)}(A) = \widetilde{\mathrm{Ind}}^V_{T,W,i,(k,l)}(A)[\dim G_V/P + d_0],$$

where d_0 is the dimension of fibers of p_0, and

(28)
$$\mathrm{Res}^V_{T,W,i,(k,l)}(A) = \widetilde{\mathrm{Res}}^V_{T,W,i,(k,l)}(A)[d_0 - \dim G_V/P].$$

Lemma 8. *Let $\iota_2 : E_{W,i,\leq(k,l)} \to E_W$ be the closed embedding, then for any $A \in \mathcal{P}^f_{W,i,(k,l)}$,*

$$\widetilde{\mathrm{Ind}}^V_{T,W}(\iota_{2!}A) = \iota_{1!}\widetilde{\mathrm{Ind}}^V_{T,W,i,(k,l)}(A).$$

Proof. Consider the following diagram:

$$\begin{tikzcd} E_{W,i,\leq(k,l)} \arrow[d, "\iota_2"'] & G_V \times^U E_{W,i,\leq(k,l)} \arrow[l, "p_1'"'] \arrow[r, "p_2'"] \arrow[d, "\iota_3"'] \arrow[l, phantom, "\boxed{1}", shift right=3ex] & G_V \times^P E_{W,i,\leq(k,l)} \arrow[r, "p_3'"] \arrow[d, "\iota_4"'] & E_{V,i,\leq(k,l)} \arrow[d, "\iota_1"] \\ E_W & G_V \times^U E_W \arrow[l, "p_1"] \arrow[r, "p_2"'] & G_V \times^P E_W \arrow[r, "p_3"'] & E_V. \end{tikzcd} \tag{29}$$

Since the squares $\boxed{1}$ and $\boxed{2}$ are cartesian squares and $\boxed{3}$ is commutative, we have

$$p_1^*\iota_{2!}A = \iota_{3!}p_1'^*A = \iota_{3!}p_2'^*(\widetilde{A}) = p_2^*\iota_{4!}(\widetilde{A}).$$

Here $\widetilde{A}$ is the unique complex such that $p_1'^*A = p_2'^*\widetilde{A}$ by Property 2.3(3).

Hence $\iota_{4!}\widetilde{A} = p_{2\flat}p_1^*\iota_{2!}A$. Therefore,

$$\widetilde{\mathrm{Ind}}^V_{T,W}(\iota_{2!}A) = p_{3!}\iota_{4!}\widetilde{A} = \iota_{1!}p'_{3!}\widetilde{A} = \iota_{1!}\widetilde{\mathrm{Ind}}^V_{T,W,i,(k,l)}(A).$$

□

Corollary 6. *For any $A \in \mathcal{P}^f_{W,i,(k,l)}$,*

$$\mathrm{Ind}^V_{T,W}(\iota_{2!}A) = \iota_{1!}\,\mathrm{Ind}^V_{T,W,i,(k,l)}(A)[N],$$

where $N = (n-1)\sum_i \mathrm{Rank}(T_i)\mathrm{Rank}(W_i) + n\sum_{h\in H}\mathrm{Rank}(T_{h'})\mathrm{Rank}(W_{h''}) - d_0$.

Proof. By Lemma 8, (19) and (27). □

Lemma 9. *Let $b : Y \to X$ be a fiber bundle with d dimensional connected smooth irreducible fiber. If $B = b^*A$ is a perverse sheaf on Y, then $b_!B[d]$ is a perverse sheaf on X.*

Proof. By the definition of perverse sheaves, $B \in \mathcal{D}^{\geq 0}(Y)\bigcap\mathcal{D}^{\leq 0}(Y)$. Then $b_!B \in \mathcal{D}^{\leq d}(X)$. i.e. $b_!B[d] \in \mathcal{D}^{\leq 0}(X)$.

On the other hand, by Lemma 3,

$$\mathbb{D}(b_!B[d]) = \mathbb{D}(b_!B)[-d] = (b_!\mathbb{D}B)[d] \in \mathcal{D}^{\leq d}(X)[d] = \mathcal{D}^{\leq 0}(X).$$

This proves the lemma. □

Proposition 10. (1) *Let $A \in \mathcal{P}^{f0}_{W,i,(k,l)}$. Then $H^n\,\mathrm{Ind}^V_{T,W,i,(k,l)}(A) \in \mathcal{P}^{f1}_{V,i,(k,l)}$ if $n \neq 0$, and $H^0\,\mathrm{Ind}^V_{T,W,i,(k,l)}(A) \in \mathcal{P}^{f0}_{V,i,(k,l)}$ So one can define a functor*

$$\begin{aligned} \xi : \mathcal{P}^{f0}_{W,i,(k,l)} &\to \mathcal{P}^{f0}_{V,i,(k,l)} \\ A &\mapsto (H^0\,\mathrm{Ind}^V_{T,W,i,(k,l)}(A))^0. \end{aligned}$$

(2) *Let* $B \in \mathcal{P}^{f0}_{V,i,(k,l)}$. *Then* $H^n \operatorname{Res}^V_{T,W,i,(k,l)}(B) \in \mathcal{P}^{f1}_{W,i,(k,l)}$ *if* $n \neq 0$ *and* $H^0 \operatorname{Res}^V_{T,W,i,(k,l)}(B) \in \mathcal{P}^{f0}_{W,i,(k,l)}$. *So one can define a functor*

$$\rho : \mathcal{P}^{f0}_{V,i,(k,l)} \to \mathcal{P}^{f0}_{W,i,(k,l)}$$
$$B \mapsto (H^0 \operatorname{Res}^V_{T,W,i,(k,l)}(B))^0.$$

(3) *The functors* $\xi : \mathcal{P}^{f0}_{W,i,(k,l)} \to \mathcal{P}^{f0}_{V,i,(k,l)}$ *and* $\rho : \mathcal{P}^{f0}_{V,i,(k,l)} \to \mathcal{P}^{f0}_{W,i,(k,l)}$ *give an equivalence of categories* $\mathcal{P}^{f0}_{V,i,(k,l)}$ *and* $\mathcal{P}^{f0}_{W,i,(k,l)}$.

Proof. The proof is based on Lusztig's idea for proving Proposition 9.3.3 in [23]. Consider the following diagram,

(30)
$$\begin{tikzcd}
G_V \times^P E_{W,i,(k,l)} \arrow[r, "p_0"] \arrow[d, "j_0"] & E_{V,i,(k,l)} \arrow[d, "j"] & E_{W,i,(k,l)} \arrow[l, "\iota_0"'] \arrow[d, "m"] \\
G_V \times^P E_{W,i,\leq(k,l)} \arrow[r, "p'_3"] & E_{V,i,\leq(k,l)} & E_{W,i,\leq(k,l)}. \arrow[l, "\iota'"']
\end{tikzcd}$$

Here ι_0, ι', j, j_0 and m are all inclusions, both squares are cartesian squares. Additionally, both j and m are open embeddings.

(1) For any $A \in \mathcal{P}^{f0}_{W,i,(k,l)}$,

$$j^* \widetilde{\operatorname{Ind}}^V_{T,W,I',\eta} A = j^* p'_{3!} p'_{2\flat} p'^*_1(A) = p_{0!} j_0^* p'_{2\flat} p'^*_1(A).$$

By Property 2.3(3), $p'_{2\flat} p'^*_1(A)[\dim(G_V/P)]$ is a perverse sheaf. j_0 is an open embedding, so $j_0^* p'_{2\flat} p'^*_1(A)[\dim(G_V/P)]$ is a perverse sheaf. Moreover $j_0^* p'_{2\flat} p'^*_1(A)[\dim(G_V/P)]$ is a G_V-equivariant perverse sheaf.

We claim that $j_0^* p'_{2\flat} p'^*_1(A) = p_0^* \iota_{0*} m^* A$.

By the following commutative diagram,

$$\begin{tikzcd}
E_{W,i,\leq(k,l)} & G_V \times^U E_{W,i,\leq(k,l)} \arrow[l, "p'_1"'] \arrow[r, "p'_2"] & G_V \times^P E_{W,i,\leq(k,l)} \\
E_{W,i,(k,l)} \arrow[u, "m"] & G_V \times^U E_{W,i,(k,l)} \arrow[l, "p''_1"'] \arrow[r, "p''_2"] \arrow[u, "j_1"] & G_V \times^P E_{W,i,(k,l)} \arrow[u, "j_0"]
\end{tikzcd}$$

we have

(31)
$$j_0^* p'_{2\flat} p'^*_1 A = p''_{2\flat} j_1^* p'^*_1 A = p''_{2\flat} p''^*_1 m^* A.$$

We next consider the following commutative diagram,

(32)
$$\begin{tikzcd}
& G_V \times^U E_{W,i,(k,l)} \arrow[dl, "p"'] \arrow[dr, "u"] & \\
E_V & G_V \times E_{W,i,(k,l)} \arrow[l, "q_1"'] \arrow[r, "u_1"] \arrow[u, "\pi"] \arrow[d, "\iota"] & E_V \\
& G_V \times E_{V,i,(k,l)} \arrow[ul, "q_2"] \arrow[ur, "u_2"'] &
\end{tikzcd}$$

Here p (resp. q_1, q_2) is the projection map sending (g, x) to $\iota_0(x)$ (resp. $\iota_0(x), x$) and u (resp. u_1, u_2) is the G_V-action map sending (g, x) to $g\iota_0(x)$ (resp. $g\iota_0(x), gx$). π is the quotient map and ι is the embedding. p is well-defined since U acts on E_W trivially in this case.

For any G_V-equivariant complex K, $q_2^*K = u_2^*K$. Then

$$q_1^*K = \iota^* q_2^*K = \iota^* u_2^*K = u_1^*K.$$

Therefore $\pi^*p^*K = \pi^*u^*K$ which implies $p^*K = u^*K$ since π is a principle U-bundle.

Now consider the following diagram,

(33)
$$\begin{tikzcd}
E_{W,i,(k,l)} \arrow[d, "\iota_0"] & G_V \times^U E_{W,i,(k,l)} \arrow[l, "p_1''"'] \arrow[d, "p_2''"] \arrow[dl, "p", "u"'] \\
E_{V,i,(k,l)} & G_V \times^P E_{W,i,(k,l)} \arrow[l, "p_0"']
\end{tikzcd}$$

where $p_1''(g, x) = x$; $u(g, x) = g\iota_0(x)$ and $p(g, x) = \iota_0(x)$. By commutativity, for any G_V-equivariant complex K, we have

(34)
$$p_0^*K = p''_{2\flat} u^*K = p''_{2\flat} p^*K = p''_{2\flat} p_1''^* \iota_0^* K$$

Since $\iota_0 : E_{W,i,(k,l)} \to E_{V,i,(k,l)}$ is the inclusion of a locally closed subset, by [24], we have

(35)
$$\iota_0^* \iota_{0*} m^*A = m^*A$$

By (31), (34) and (35), we have

$$j_0^* p'_{2\flat} p_1'^* A = p''_{2\flat} p_1''^* \iota_0^* \iota_{0*} m^*A = p_0^* \iota_{0*} m^*A.$$

This proves the claim.

Therefore, by Lemma 9, $j^*\widetilde{\mathrm{Ind}}^V_{T,W,I',\eta} A[\dim(G_V/P) + d_0]$ is a perverse sheaf on $E_{V,i,(k,l)}$. Since j is an open embedding, j^* is exact. if $n \neq 0$,

$$j^*(H^n(\widetilde{\mathrm{Ind}}^V_{T,W,i,(k,l)} A[\dim(G_V/P) + d_0])) = H^n(j^*\widetilde{\mathrm{Ind}}^V_{T,W,i,(k,l)} A[\dim(G_V/P) + d_0]) = 0.$$

i.e.

$$j^*(H^n(\mathrm{Ind}^V_{T,W,i,(k,l)} A)) = 0.$$

Therefore, the support of $H^n \operatorname{Ind}^V_{T,W,i,(k,l)} A$ is disjoint from $E_{V,i,(k,l)}$.

(2) For any $B \in \mathcal{P}^{f0}_{V,i,(k,l)}$, j^*B is a perverse sheaf since j is an open embedding.

We claim that $\iota_0^* j^* B[d_0 - \dim(G_V/P)]$ is a perverse sheaf on $E_{W,i,(k,l)}$.

In Diagram (33), p_1'' is a fiber bundle of relative dimension $\dim(G_V/U)$ since $\mathrm{Supp}(T) = \{i\}$ and U acts on E_W trivially.

By the commutativity, we have

$$p_2''^* p_0^* j^* B = u^* j^* B = p^* j^* B = p_1''^* \iota_0^* j^* B.$$

Then

$$\iota_0^* j^* B[d_0 - \dim(G_V/P)] = p_{1!}'' p_2''^* p_0^* j^* B[d_0 + \dim(G_V/U) + \dim(G_V/P)].$$

By Lemma 7, $p_2''^* p_0^* j^* B[d_0 + \dim(P/U)]$ is a perverse sheaf. From Lemma 9, $\iota_0^* j^* B[d_0 - \dim(G_V/P)]$ is a perverse sheaf on $E_{W,i,(k,l)}$. This proves the claim.

Since right hand square in Diagram (30) is commutative,

$$m^* \iota'^* B[d_0 - \dim(G_V/P)] = \iota_0^* j^* B[d_0 - \dim(G_V/P)]$$

which is a perverse sheaf. Since m is open embedding,

$$m^*(H^n \iota'^* B[d_0 - \dim(G_V/P)]) = H^n(m^* \iota'^* B[d_0 - \dim(G_V/P)]).$$

If $n \neq 0$, support of $H^n \iota'^* B[d_0 - \dim(G_V/P)]$ is disjoint form $E_{V,i,(k,l)}$.

(3) From the proof of (1), we have

$$j^* \xi(A) = j^* p'_{3!} p'_{2\flat} p_1'^* A[\dim(G_V/P) + d_0] = p_{0!} j_0^* p'_{2\flat} p_1'^* A[\dim(G_V/P) + d_0].$$

Hence

$$j^*(\xi(\rho(B))) = p_{0!} j_0^* p'_{2\flat} p_1'^* \rho(B)[\dim(G_V/P) + d_0] = p_{0!} j_0^* p'_{2\flat} p_1'^* \iota'^*(B)[2d_0].$$

Consider the following diagram,

$$\begin{tikzcd}
E_{W,i,\leq(k,l)} \arrow[d, "\iota'"] & G_V \times^U E_{W,i,\leq(k,l)} \arrow[l, "p_1'"'] \arrow[d, "p_2'"] \arrow[dl, "p"', "u"] \\
E_{V,i,\leq(k,l)} & G_V \times^P E_{W,i,\leq(k,l)}. \arrow[l, "p_3'"']
\end{tikzcd}$$

Here $u(g,x) = g\iota'(x)$ and $p(g,x) = \iota_0(x)$. By the same reason as above, p is well-defined. By commutativity, we have

$$p_3'^* B = p'_{2\flat} u^* B = p'_{2\flat} p^* B = p'_{2\flat} p_1'^* \iota'^* B \tag{36}$$

From Diagram (30), $p_0^* j^* B = j_0^* p_3'^* B$. Then

$$j_0^* p'_{2\flat} p_1'^* \iota'^*(B) = j_0^* p_3'^* B = p_0^* j^* B.$$

Therefore,

$$j^*(\xi(\rho(B))) = p_{0!} p_0^* j^* B[2d_0].$$

By Lemma 7, $j^*(\xi(\rho(B))) = j^* B$.

Since $B \in \mathcal{P}^{f0}_{V,i,(k,l)}$ and $E_{V,i,(k,l)}$ is open in $E_{V,i,\leq(k,l)}$, we have $\xi(\rho(B)) = B$.

On the other hand, from the proof of (1), we have

$$m^*(\rho(B)) = \iota_0^* j^* B[d_0 - \dim(G_V/P)].$$

Hence,

$$m^*(\rho(\xi(A))) = \iota_0^* j^* \xi(A)[d_0 - \dim(G_V/P)] = \iota_0^* p_{0!} j_0^* p'_{2\flat} p_1'^* A[2d_0].$$

From the proof of (1), we have

$$m^*(\rho(\xi(A))) = \iota_0^* p_{0!} p_0^* \iota_{0*} m^* A[2d_0] = \iota_0^* \iota_{0*} m^* A = m^* A.$$

Since $A \in \mathcal{P}^{f0}_{W,i,(k,l)}$ and $E_{W,i,(k,l)}$ is open in $E_{W,i,\leq(k,l)}$, we have $\rho(\xi(A)) = A$. $\square$

Remark 7. By Proposition 5, if $E_W \simeq F$, then $\mathrm{Res}^V_{T,W,i,(k,l)}$ send any element of $\mathcal{P}^f_{V,i,(k,l)}$ into $\mathcal{P}^f_{W,i,(k,l)}$, and the map ρ is well defined.

Let v be an indeterminate and $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. Let $\mathcal{M}_V$ be the Grothendieck group of the category which consists of all direct sums of $L_{V,\underline{i},\underline{k}}$ for various $(\underline{i}, \underline{k})$ and their shifts. Define an $\mathcal{A}$-action on $\mathcal{M}_V$ by $v^n \cdot L = L[n]$. Then $\mathcal{M}_V$ is an $\mathcal{A}$-module generated by $L_{V,\underline{i},\underline{k}}$. Let $\mathcal{K}_V$ be the Grothendieck group of category $\mathcal{Q}^f_V$ (see Section 4.1). Then under the same $\mathcal{A}$-action, $\mathcal{K}_V$ is an $\mathcal{A}$-module generated by the simple perverse sheaves in $\mathcal{P}^f_V$.

Theorem 3. *$\mathcal{M}_V \simeq \mathcal{K}_V$ as an $\mathcal{A}$-module, i.e., $\{L_{V,\underline{i},\underline{k}} \mid \forall(\underline{i},\underline{k})\}$ are the additive generators of $\mathcal{K}_V$.*

Proof. Clearly, $\mathcal{M}_V \subseteq \mathcal{K}_V$ since $L_{V,\underline{i},\underline{k}}$ is a direct sum of simple perverse sheaves in $\mathcal{P}^f_V$. By abuse of notation, we will denote by the same B the isomorphism class of B in $\mathcal{K}_V$ (resp. $\mathcal{M}_V$). One only needs to show $B \in \mathcal{M}_V$ for any simple perverse sheaf $B \in \mathcal{P}^f_V$.

We first use induction on $Td(V) := \sum_{i\in I} \mathrm{Rank}(V_i)$. If $V = 0$, then $E_V = \{\mathrm{pt}\}$. So there is only one simple perverse sheaf and thus the theorem is true. Now assume the theorem is true for any I-graded proper R-submodule W of V. We want to show the theorem is true for V.

Suppose B is a simple direct summand of $L_{V,\underline{j},\underline{l}}$, where $(\underline{j},\underline{l}) = ((i,k),(\underline{j}',\underline{l}'))$. Recall $L_{V,\underline{j},\underline{l}} = \mathrm{Ind}^V_{T,W}(L_{T,i,k} \boxtimes L_{W,\underline{j}',\underline{l}'})$. Since $\mathrm{Supp}(T) = \{i\}$, $E_T = \{\mathrm{pt}\}$. We will simply write $L_{T,i,k} \boxtimes L_{W,\underline{j}',\underline{l}'}$ as $L_{W,\underline{j}',\underline{l}'}$. By Fourier-Deligne transform, we can further assume i is a sink. By the definition of $\mathrm{Ind}^V_{T,W}$, we have

$$\mathrm{Supp}(B) \subset \mathrm{Supp}(L_{V,\underline{j},\underline{l}}) \subset \mathrm{Supp}(\mathrm{Ind}^V_{T,W}(L_{W,\underline{j}',\underline{l}'})) \subset E_{V,i,\leq(m,l)}$$

for some $(m,l) \in \mathbb{N} \times \mathbb{N}$. By (22), we can choose maximal (m,l) such that $\mathrm{Supp}(B) \subset E_{V,i,\leq(m,l)}$ and $\mathrm{Supp}(B)$ meets $E_{V,i,(m,l)}$.

Recall $\iota_1 : E_{V,i,\leq(m,l)} \hookrightarrow E_V$ is the closed embedding. Since $\mathrm{Supp}(B) \subset E_{V,i,\leq(m,l)}$, by the definition of $\mathcal{P}^{f0}_{V,i,(m,l)}$, we have $\iota_1^* B \in \mathcal{P}^{f0}_{V,i,(m,l)}$. Let $\rho(\iota_1^* B) = A$. By Proposition 10, $\xi(A) = \iota_1^* B$. i.e.,

$$\mathrm{Ind}^V_{T,W,i,m} A = \iota_1^* B \oplus (\oplus_j C_j[j]) \tag{37}$$

for some C_j satisfying $\mathrm{Supp}(C_j) \subset E_{V,i,\leq(m,l)}$ and disjoint with $E_{V,i,(m,l)}$. Therefore $\mathrm{Supp}(C_j) \subset E_{V,i,\leq(r,s)}$ with $(r,s) < (m,l)$. By applying $\iota_{1!}$ to (37) and Corollary 6,

$$\mathrm{Ind}^V_{T,W} \iota_{2!} A = \iota_{1!} \mathrm{Ind}^V_{T,W,i,(m,l)} A[N] = \iota_{1!} \iota_1^* B[N] \oplus (\oplus_j \iota_{1!} C_j[N+j]),$$

where $\iota_2 : E_{W,i,\leq(m,l)} \hookrightarrow E_W$ is the closed embedding and N is defined in Corollary 6. Since $\mathrm{Supp}(B) \subset E_{V,i,\leq(m,l)}$,

$$\iota_{1!} \iota_1^* B = B|_{E_{V,i,\leq(m,l)}} = B.$$

Now we want to show $\iota_{2!}A \in \mathcal{M}_W$.

By the definition of ρ, in fact, $\iota_{2!}A$ is a direct summand of $\iota_{2!}\iota'^*\iota_1^*B[d_0 - \dim G_V/P]$. Applying base change formula to the following cartesian square,

$$\begin{tikzcd} E_{W,i,\leq(m,l)} \arrow[r, "\iota'"] \arrow[d, "\iota_2"] & E_{V,i,\leq(m,l)} \arrow[d, "\iota_1"] \\ E_W \arrow[r, "\iota"] & E_V, \end{tikzcd}$$

we have

$$\iota_{2!}\iota'^*\iota_1^*B = \iota^*\iota_{1!}\iota_1^*B = \iota^*B = \widetilde{\mathrm{Res}}_{T,W}^V B.$$

By Proposition 4, $\iota_{2!}A \in \mathcal{K}_W$. Since $Td(W) < Td(V)$, by the assumption, $\iota_{2!}A \in \mathcal{M}_W$.

Therefore, $\mathrm{Ind}_{T,W}^V \iota_{2!}A \in \mathcal{M}_V$ by Corollary 2.

To show $B \in \mathcal{M}_V$, it is enough to show $\iota_{1!}C_j \in \mathcal{M}_V$. Since $\iota_{1!}C_j$ is a direct summand of $\mathrm{Ind}_{T,W}^V \iota_{2!}A$ and $\iota_{2!}A \in \mathcal{Q}_W^f$, by Proposition 6, $\iota_{1!}C_j \in \mathcal{Q}_V^f$.

To apply induction on (m,l), it is enough to show $C_j[j] \in \mathcal{M}_V$ if $\mathrm{Supp}(C_j[j]) \subset E_{V,i,\leq(0,l)}$. By a similar argument as above, there exists $K \in \mathcal{P}_{W,i,\leq(0,l)}^{f0}$ such that

$$\mathrm{Ind}_{T,W}^V(\iota'_{2!}K) = \iota'_{1!}\iota_1'^*C_j[M+j] = C_j[M+j]$$

for some M, where ι'_1 and ι'_2 are the embedding maps. By induction on $Td(V)$, $C_j[j] \in \mathcal{M}_V$ since $\iota'_{2!}K \in \mathcal{M}_W$ as we have shown above.

The theorem follows from the induction on (m,l). $\square$

5. Geometric approach to Hall algebras and quantum generalization Kac-Moody algebras

5.1. **The algebra ($\mathcal{K}$, Ind).** Recall the dimension vector of an I-graded free R-module V is defined as $|V| := (\mathrm{Rank}(V_i))_{i\in I} \in \mathbb{N}I$. It is important to notice that, given two different I-graded free R-modules V and V' with the same dimension vector, $\mathcal{K}_V \simeq \mathcal{K}_{V'}$ since E_V and $E_{V'}$ are isomorphism spaces. So one may denote $\mathcal{K}_V$ by $\mathcal{K}_{|V|}$. Moreover, the functors $\mathrm{Ind}_{T,W}^V$ and $\mathrm{Res}_{T,W}^V$ can be rewritten as $\mathrm{Ind}_{|T|,|W|}^{|T|+|W|}$ and $\mathrm{Res}_{|T|,|W|}^{|T|+|W|}$ respectively. Now let $\mathcal{K} = \oplus_{|V|\in\mathbb{N}I}\mathcal{K}_{|V|}$. Define multiplication as follows.

$$\mathrm{Ind} : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$$

$$(A,B) \mapsto \mathrm{Ind}_{|T|,|W|}^{|T|+|W|}(A \otimes B)$$

for homogenous elements A, B with $A \in \mathcal{K}_{|T|}$ and $B \in \mathcal{K}_{|W|}$.

Theorem 4. (1) *$\mathcal{K}$ equipped the multiplication* Ind *is an I-graded associated $\mathcal{A}$-algebra.*

(2) $\{L_{V,\underline{i},\underline{k}} \mid$ for all V and $(\underline{i},\underline{k})\}$ *contains an $\mathcal{A}$-basis of $\mathcal{K}$. This basis is called a monomial basis.*

(3) *All simple perverse sheaves in $\mathcal{P}_V^f$ for various V form an $\mathcal{A}$-basis of $\mathcal{K}$. This basis is called the canonical basis.*

Proof. (1) follows from Theorem 3, Corollary 2 and additivity of Ind. (2) follows from Theorem 3. (3) follows from the definition of $\mathcal{K}$. $\square$

In the rest of this section we will give the relation among the Hall algebra $C\mathcal{H}_R$ (see Section 3), the algebra $\mathcal{K}$, and the quantum generalized Kac-Moody algebra.

5.2. **Relation between $\mathcal{K}$ and U_v^-.**

5.2.1. Let I be a countable index set. A simply laced *generalized root datum* (see [10]) is a matrix $A = (a_{ij})_{i,j\in I}$ satisfying the following conditions:

(i) $a_{ii} \in \{2, 0, -2, -4, \cdots\}$, and
(ii) $a_{ij} = a_{ji} \in \mathbb{Z}_{\leq 0}$.

Such a matrix is a special case of Borcherds-Cartan matrix. Let $I^{re} = \{i \in I \mid a_{ii} = 2\}$ and $I^{im} = I \setminus I^{re}$. A collection of positive integers $m = (m_i)_{i\in I}$ with $m_i = 1$ whenever $i \in I^{re}$ is called the charge of A.

5.2.2. The *quantum generalized Kac-Moody algebra* (see [10]) associated with (A, m) is the $\mathbb{Q}(v)$-algebra $U_v(\mathfrak{g}_{A,m})$ generated by the elements K_i, K_i^{-1}, $E_{i,k}$, and $F_{i,k}$ for $i \in I, k = 1, \cdots, m_i$ subject to the following relations:

$$K_iK_i^{-1} = K_i^{-1}K_i = 1,\ K_iK_j = K_jK_i, \tag{38}$$

$$K_iE_{jk}K_i^{-1} = v^{a_{ij}}E_{jk},\ K_iF_{jk}K_i^{-1} = v^{-a_{ij}}F_{jk}, \tag{39}$$

$$E_{ik}F_{jl} - F_{jl}E_{ik} = \delta_{lk}\delta ij\frac{K_i - K_i^{-1}}{v - v^{-1}}, \tag{40}$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1 - a_{ij} \\ n \end{bmatrix} E_{ik}^{1-a_{ij}-n}E_{jl}E_{ik}^n = 0, \forall i \in I^{re}, j \in I, i \neq j, \tag{41}$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1 - a_{ij} \\ n \end{bmatrix} F_{ik}^{1-a_{ij}-n}F_{jl}F_{ik}^n = 0, \forall i \in I^{re}, j \in I, i \neq j, and \tag{42}$$

$$E_{ik}E_{jl} - E_{jl}E_{ik} = F_{ik}F_{jl} - F_{jl}F_{ik} = 0,\ \text{if}\ a_{ij} = 0. \tag{43}$$

Here $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[n-k]![k]!}$, $[n]! = \prod_{i=1}^n [i]$, and $[n] = \frac{v^n - v^{-n}}{v - v^{-1}}$.

In this paper, we only consider the case in which all $m_i = 1$ and all indices are in I^{im}.

5.2.3. Define a bilinear form on $\mathcal{K}$ as follows,

$$(A, B)_{\mathcal{K}} = \sum_j d_j(E_V, G_V; A, B)v^{-j}.$$

Proposition 11. *The bilinear form $(-,-)_{\mathcal{K}}$ defined above is non-degenerate.*

Proof. Firstly, by the properties of $d_j(E, G; A, B)$ (see Section 4.4), this is a bilinear form. Secondly, by Theorem 4, all simple perverse sheaves in $\mathcal{P}_V^f$ for various V form a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathcal{K}$. So, for any $A \in \mathcal{K}$, A can be written as $A = \sum_K c_K K$ for $c_K \in \mathbb{Z}[v, v^{-1}]$. Now for any $A \in \mathcal{K}$, let B is a direct summand of A. Then by Property (3) of $d_j(E, G; A.B)$ in Section 4.4, $(A, B)_{\mathcal{K}} = c_B \neq 0$. Hence the bilinear form is non-degenerate. $\square$

Let U_v^- be the $\mathbb{Z}[v, v^{-1}]$-subalgebra of $U_v(\mathfrak{g}_{A,m})$ generated by all $F_{i,k}$ with $k = 1, \cdots, m_i$. Because we only consider the case that the charge $m_i = 1$ for all $i \in I$, there is only one generator $F_{i,1}$ for each i, which we will simply denote by F_i. It is clear that U_v^- only subjects to one relation, namely $F_iF_j = F_jF_i$ if $a_{ij} = 0$.

Define a multiplication of $U_v^- \otimes U_v^-$ as

$$(A \otimes B)(C \otimes D) := v^{-n(|B|,|C|)}(AC) \otimes (BD),$$

where $|B|$ is the grading of B when B is a homogeneous element.

Let $\mathfrak{F}$ be the free algebra generated by $\{F_i \mid i \in I\}$. Let $r' : \mathfrak{F} \to U_v^- \otimes U_v^-$ be the algebra homomorphism sending F_i to $F_i \otimes 1 + 1 \otimes F_i$. Since $r'(F_iF_j) = r'(F_jF_i)$ if $a_{ij} = 0$, the map r' induces an algebra homomorphism $r : U_v^- \to U_v^- \otimes U_v^-$. This gives a coalgebra structure on U_v^-.

Define a map

$$\begin{aligned} f : U_v^- &\to \mathcal{K} \\ F_i &\mapsto L_{i,1}. \end{aligned}$$

It is easy to check that $f(F_iF_j) = f(F_jF_i)$. So this map can be extended to an algebra homomorphism. In addition, f preserves the grading, where the grading on $\mathcal{K}$ is defined in (**??**). Now define a bilinear form $(-,-)_U$ on U_v^- as $(A, B)_U := (f(A), f(B))_{\mathcal{K}}$.

Theorem 5. *$\mathrm{Ker}(f) = Rad(-,-)_U =: \mathcal{I}_1$, so $U_v^-/\mathcal{I}_1 \simeq \mathcal{K}$.*

Proof. Obviously, $\mathrm{Ker}(f) \subset \mathcal{I}_1$.

Let us pick any $x \in \mathcal{I}_1$. Then for any $y \in \mathcal{K}$, there exists $z \in U_v^-$ such that $f(z) = y$ due to the fact that f is surjective. Therefore,

$$0 = (x, z)_U = (f(x), f(z))_{\mathcal{K}} = (f(x), y)_{\mathcal{K}}.$$

This means $f(x) \in Rad(-,-)_{\mathcal{K}}$. Since the bilinear form $(-,-)_{\mathcal{K}}$ is non-degenerate, $f(x) = 0$. i.e., $x \in \mathrm{Ker}(f)$. Hence $U_v^-/\mathcal{I}_1 \simeq \mathcal{K}$. ◻

5.3. **Relation between $\mathcal{K}$ and $C\mathcal{H}_R$.** Let $\mathcal{H}(R\Gamma)^*$ be the dual Hall algebra of $\mathcal{H}(R\Gamma)$, i.e., $\mathcal{H}(R\Gamma)^* = \oplus_\nu \mathcal{H}(R\Gamma)^*_\nu$. Here $\mathcal{H}(R\Gamma)^*_\nu$ is the set of all $\mathbb{C}$-valued functions on the set of isomorphism classes of all representations M of Γ over R with dimension vector $|M| = \nu$. The multiplication on $\mathcal{H}(R\Gamma)^*$ is defined as follows:

$$(f_1 \cdot f_2)(E) = \sum_{N \subset E} f_1(E/N) f_2(N).$$

See [13] for more information. Let $C\mathcal{H}_R^*$ be the subalgebra of $\mathcal{H}(R\Gamma)^*$ generated by $\delta_{S_i}, \forall i \in I$, where δ_{S_i} is the characteristic function of S_i. i.e.

$$\delta_{S_i}(x) = \begin{cases} 1 & \text{if } x = S_i \\ 0 & \text{others.} \end{cases}$$

By the following formular, $C\mathcal{H}_R^*$ is isomorphic to the algebra $C\mathcal{H}_R$.

$$(\delta_M \cdot \delta_N)(E) = \#\{L \subset E \mid L \simeq N, E/L \simeq M\} = F^E_{M,N}.$$

Now define

$$\chi : \mathcal{K} \to C\mathcal{H}_R^*$$
$$A \mapsto \chi_A,$$

where $\chi_A(x) = Tr(Fr : A^{\cdot}_x \to A^{\cdot}_x)$ (see Section 2.5).

Lemma 10 (Theorem 4.1(b) in [13])**.** *$\chi : \mathcal{K} \to C\mathcal{H}_R^*$ is an algebra homomorphism.*

Theorem 6. *χ is a surjective algebra homomorphism.*

Proof. By Lemma 10, it is enough to show $\chi_{L_i} = \delta_{S_i}$ for any δ_{S_i}. In fact,

$$\chi_{L_i}(x) = \chi_{\pi_{i!}\mathbf{1}}(x) = \sum_{y \in \pi_i^{-1}(x)} \chi_{\mathbf{1}}(y) = \chi_{\mathbf{1}}(\pi_i^{-1}(x)) = \delta_{S_i}.$$

Here π_i is the obvious projection map. The penultimate equality is true because both E_V and $\widetilde{\mathcal{F}}_V$ contain a single point. □

Let us denote $\mathcal{I}_2 = \mathrm{Ker}(\chi)$. Then $C\mathcal{H}_R \simeq \mathcal{K}/\mathcal{I}_2$.

Remark 8. One may ask what the kernel $\mathcal{I}_1$ of the above map f is. For the field case, Lusztig and Ringel show this is the ideal generated by the quantum Serre relations. However, for the case where we have the local ring $R = k[t]/(t^n)$, the kernel $\mathcal{I}_1$ is much more complicated. Let's finish this paper with the following example which gives some idea of what $\mathcal{I}_1$ is.

Example 2. Fix $R = \mathbb{F}_q[t]/(t^n)$, consider quiver $A_2 : 1 \to 2$. Then $S_1 : R \to 0$ and $S_2 : 0 \to R$ are all simple objects. By computation, one has,

$S_1^2 = q^{n/2}(q^n + q^{n-1})(R^2 \to 0)$,

$S_1S_2 = q^{-n/2}[(R \xrightarrow{0} R) + (R \xrightarrow{1} R) + (R \xrightarrow{t} R) + \cdots + (R \xrightarrow{t^{n-1}} R)]$,

$S_2S_1 = R \xrightarrow{0} R$,

$S_1^2S_2 = q^{-n/2}(q^n+q^{n-1})[(R^2 \xrightarrow{0} R)+(R^2 \xrightarrow{(1,0)} R)+(R^2 \xrightarrow{(t,0)} R)+\cdots+(R^2 \xrightarrow{(t^{n-1},0)} R)]$,

$S_1S_2S_1 = (q^n+q^{n-1})(R^2 \xrightarrow{0} R)+(R^2 \xrightarrow{(1,0)} R)+q(R^2 \xrightarrow{(t,0)} R)+\cdots+q^{n-1}(R^2 \xrightarrow{(t^{n-1},0)} R)$,

and

$S_2S_1^2 = q^{n/2}(q^n + q^{n-1})(R^2 \xrightarrow{0} R)$.

There is no quantum Serre relation at this time.

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